Perspectives and Application of Fuzzy Initial Value Problems

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BY

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REG NO: 2014MSC08

UNDER THE GUIDANCE OF

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DECLARATION

I hereby certify that the work which is being presented in the report entitled "**Perspectives and Application of Fuzzy Initial Value Problems**" in partial fulfillment of the requirement for the award of the degree of Master of Science, submitted to the Department of Mathematics, Motilal Nehru National Institute of Technology, Allahabad is carried out under the supervision of Dr. Pitam Singh.

The matter embodied in this report has not been submitted by me for the award of any other degree.

DATE: (RAJKRISHNA MONDAL)

REG No-2014MSC08

Motilal Nehru National Institute of Technology

CERTIFICATE

This is to certify that the Project Report entitled **"Perspectives and Application of Fuzzy Initial Value Problems"** submitted by Rajkrishna Mondal (Reg No: 2014MSC08) in partial fulfillment of the requirements for the degree of Master of Science in Mathematics and Scientific Computing at the Department of Mathematics, Motilal Nehru National Institute of Technology, Allahabad is an authentic work carried out by him under my supervision and guidance.

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Preface

In many cases, information about the behavior of a dynamical system is uncertain. In order to obtain a more realistic (not more exact!) model, we have to take into account these uncertainties. Also, in several cases the uncertainties are not of statistical type. For example, having some linguistic information and when we cannot repeat a measurement are such cases.

Fuzzy differential equations (FDE) are a natural way to model dynamical systems under possibilistic uncertainty. Also, in modeling real world phenomena, fuzzy initial value problems (FIVP) appear naturally and not always as a fuzzified version of a crisp problem. Therefore, we do not start with a crisp equation which is fuzzified, but with a fuzzy model adequate for some real world phenomena.

Acknowledgements

I wish to express my sincere gratitude to our project Supervisor, **Dr. Pitam Singh**, Assistant Professor, Department of Mathematics, MNNIT, ALLAHABAD for his valuable guidance and motivation without which it would not have been possible to bring out this assignment in the present form.

I would like to express my special thanks to all faculty members of Department of Mathematics, MNNIT, ALLAHABAD for providing me much needed mental support, affection and encouragement in the completion this dissertation.

I cannot sum up this note without mentioning my Parents who provided me much needed support and Parental affection, without which I would have lost somewhere in this vale of afflictions.

I am also great full to all my classmates, friends and well-wishers, for providing me big support.

Above all, I am indebted to the Almighty God would have never seen its existence.

RAJKRISHNA MONDAL

CONTENTS

Fundamentals of Fuzzy Sets

In mathematics, fuzzy sets are sets whose elements have degrees of membership. Fuzzy sets were introduced by **Lotfi A. Zadeh** and **Dieter Klaua** in 1965 as an extension of the classical notion of set. At the same time, Salii (1965) defined a more general kind of structure called an *L*-relation, which he studied in an abstract algebraic context. Fuzzy relations, which are used now in different areas, such as linguistics (De Cock, Bodenhofer & Kerre 2000) decision-making (Kuzmin 1982) and clustering (Bezdek 1978), are special cases of *L*-relations when *L* is the unit interval

[0, 1].

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition — an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. In fuzzy set theory, classical bivalent sets are usually called *crisp* sets. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics.

1.1 Introductions of Fuzzy Sets

Definition of Fuzzy Set

A fuzzy set is a pair (A, μ_A) where A is a set and $\mu_A : A \to [0,1]$.

For each x the value $\mu_A(x)$ is called the grade of membership of x in (A, μ_A).For a finite set $A = \{x_1, \ldots, x_n\}$, the fuzzy set (A, μ_A) is often denoted by $\left\{ \frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2}, \ldots, \frac{\mu_A(x_n)}{x_n} \right\}$.

Let $x \in A$. Then x is called *not included* in the fuzzy set (A, μ_A) if $\mu_A(x) = 0$, x is called *fully included* if $\mu_A(x) = 1$ and x is called a fuzzy member if $0 < \mu_A(x) < 1$

Example : We consider statement "Jenny is young". At this time, the term "young" is vague. To represent the meaning of "vague" exactly, it would be necessary to define its membership function as in Fig 1.1. When we refer "young", there might be age which lies in the range [0,80] and we can account these "young age" in these scope as a continuous set. The horizontal axis shows age and the vertical one means the numerical value of membership function. The line shows possibility (value of membership function) of being contained in the fuzzy set "young". For example, if we follow the definition of "young" as in the figure, ten year-old boy may well be young. So the possibility for the "age ten" to join the fuzzy set of "young is 1. Also that of "age twenty seven" is 0.9. But we might not say young to a person who is over sixty and the possibility of this case is 0.Now we can manipulate our last sentence to "Jenny is very young". In order to be included in the set of "very young", the age should be lowered and let us think the line is moved leftward as in the figure. If we define fuzzy set as such, only the person who is under forty years old can be included in the set of "very young". Now the possibility of twenty-seven year old man to be included in this set is 0.5.

That is, if we denote A= "young" and B="very young",

$$\mu_A(27) = 0.9, \mu_B(27) = .5$$

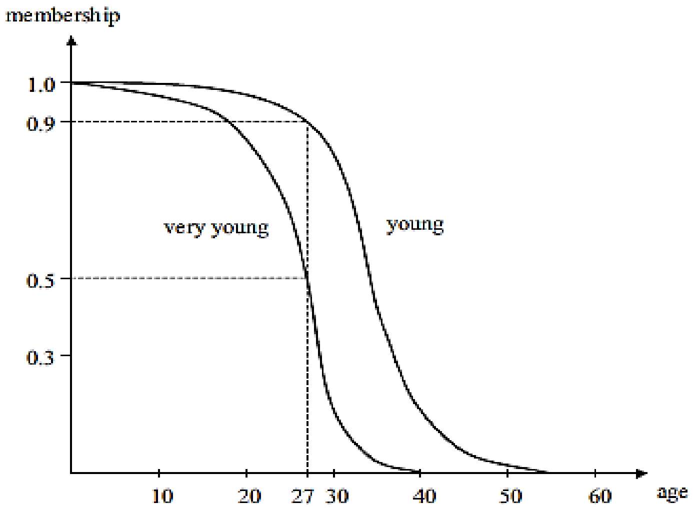


Fig. 1.1. Fuzzy sets representing "young" and "very young"

1.2 Expanding concepts of Fuzzy Set

1.2.1 Definition (Type-n Fuzzy Set) : The value of membership degree might include uncertainty. If the value of membership function is given by a fuzzy set, it is a type-2 fuzzy set. This concept can be extended up to Typen fuzzy set.

Example : Consider set A= "adult". The membership function of this set maps whole age to "youth", "manhood" and "senior. For instance, for any person x, y, and z,

$\mu_A(x)$ = "youth"

$\mu_A(y)$ = "manhood"

$\mu_A(z) = \phi$.

The values of membership for "youth" and "manhood" are also fuzzy sets ,and thus the set "adult" is a type-2 fuzzy set.The sets "youth" and "manhood" are type-1 fuzzy sets. In the same manner, if the values of membership function of "youth" and "manhood"are type-2, the set "adult" is type-3.

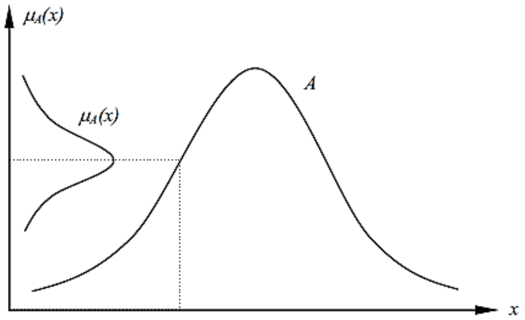


Fig. 1.2. Fuzzy Set of Type-2

1.2.2 Definition (Level-k fuzzy set): The term "level-2 set" indicates fuzzy sets whose elements are fuzzy sets (Fig 1.8). The term "level-1 set" is applicable to fuzzy sets whose elements are no fuzzy sets ordinary elements. In the same way, we can derive up to level-k fuzzy set.

Example : In the figure, there are 3 fuzzy set elements $A = \left\{ \frac{0.5}{A1}, \frac{1.0}{A2}, \frac{0.5}{A3} \right\}$

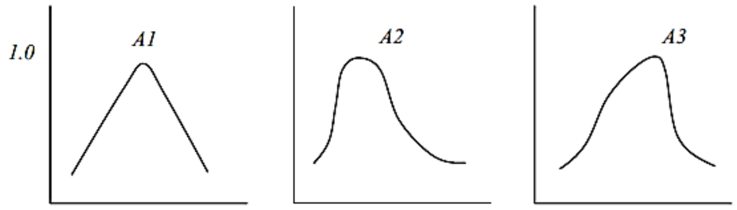


Fig. 1.3 (a) elements of level-2 fuzzy set, A1, A2,A3

$$\mu_A(A1) = 0.5$$
$$\mu_A(A2) = 1.0$$
$$\mu_A(A3) = 0.5$$

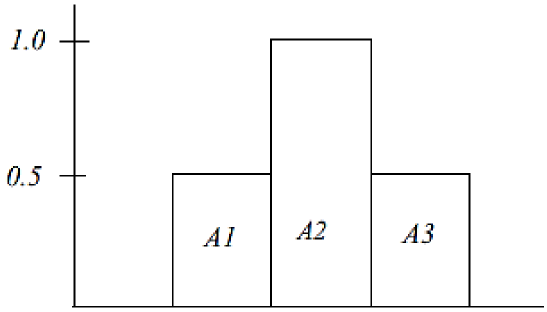


Fig. 1.3 (b) level-2 fuzzy set

Example : Consider a universal set X which is defined on the age domain.

X = {5, 15, 25, 35, 45, 55, 65, 75, 85}

We can define fuzzy sets such as "infant", "young", "adult" and "senior" in X. The possibilities of each element of x to be in those four fuzzy sets are in Table

age(element)	infant	young	adult	senior
5	0	0	0	0
15	0	0.2	0.1	0
25	0	1	0.9	0
35	0	0.8	1	0
45	0	0.4	1	0.1
55	0	0.1	1	0.2
65	0	0	1	0.6
75	0	0	1	1
85	0	0	1	1

Example of fuzzy set

1.3 Some kind of Fuzzy Set:

1.3.1 α-Cut Set:

The α-cut set A_α is made up of members whose membership is not less than α.

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$$

note that α is arbitrary. This α-cut set is a crisp set.

Strong α-Cut: $A_{+\alpha} = \{x \in X \mid \mu_A(x) > \alpha\}$

Example: In the example 1.4,

$$Young_{0.2} = \{15, 25, 35, 45\}$$

$$Young_{+0.2} = \{25, 35, 45\}$$

1.3.2 Level set :

The value α which explicitly shows the value of the membership function, is in the range of [0,1]. The "level set" is obtained by the α's. That is,

$$\Lambda_A = \{\alpha \mid \mu_A(x) = \alpha,\ \alpha \geq 0,\ x \in X\}$$

Example: The level set of the above fuzzy set "young" is,

$$\Lambda_A = \{0, 0.1, 0.2, 0.4, 0.8, 1.0\}$$

1.3.3 Fuzzy Subset :

A Fuzzy set A is said to be fuzzy subset of B if $\forall x \in A\ ,\ \mu_A(x) \leq \mu_B(x)$.

1.3.4 Hight of the Fuzzy Set :

$$h(A) = \sup_{x \in X}\{\mu_A(x)\},\ \text{for Continuous}$$

$$h(A) = \sup_{x \in X}\{\mu_A(x)\},\ \text{for Discrete}$$

1.3.5 Normal Fuzzy Set :

A Fuzzy Set A is said to be normal if $h(A) = 1$

1.3.6 Core of Fuzzy Set :

$$core(A) = \{x \in X \mid \mu_A(x) = 1\}$$

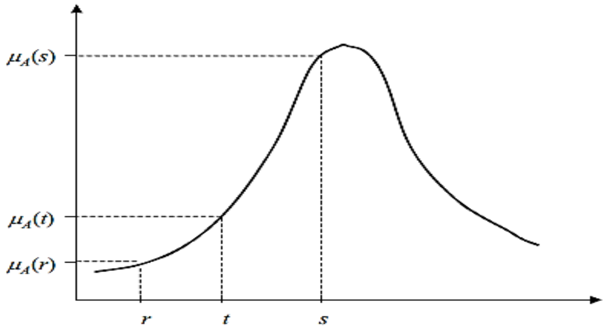
1.3.7 Support of the Fuzzy Set :

$$supp(A) = \{x \in X \mid \mu_A(x) > 0\}$$

1.3.8 Definition (Convex fuzzy set):

Assuming universal set X is defined in ndimensional Euclidean Vector space $\mathbb{R}^n$. If all the α- cut sets are convex,the fuzzy set with these α-cut sets is convex. In other words, if a relation

$$\mu_A(t) \geq \min\{\mu_A(r), \mu_A(s)\} \text{ where } t = \lambda r + (1-\lambda)s;\ r, s \in \mathbb{R}^n$$



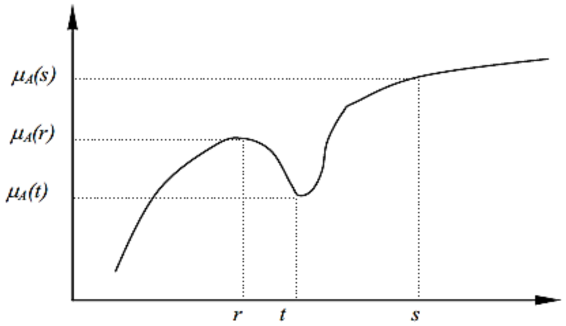


Fig 1.4 a) Convex Fuzzy Set $\mu_A(t) \geq \mu_A(r)$ **Fig 1.4** b) Non-Convex Fuzzy Set $\mu_A(t) \neq> \mu_A(r)$

1.4 The Magnitude of Fuzzy Set:

In order to show the magnitude of fuzzy set, there are three ways of measuring the cardinality of fuzzy set.

1.4.1)scalar cardinality:

we can derive magnitude by summing up the membership degrees.

$$|A| = \sum_{x \in X} \mu_A(x)$$

The magnitude of fuzzy set "senior" (in the **example 1.4**) is,

|senior| = 0.1 + 0.2 + 0.6 + 1 + 1 = 2.9

1.4.2)relative cardinality:

Comparing the magnitude of fuzzy set A with that of universal set X can be an idea.

$$\|A\| = \frac{|A|}{|X|}$$

In the case of "senior",|senior| = 2.9, |X| = 9,||senior|| = 2.9/9 = 0.32

1.4.3)Fuzzy cardinality:

Let's try to get α-cut set (crisp set) A_α , of A. The number of elements is $|A_\alpha|$. In other words, the possibility for number of elements in A to be $|A_\alpha|$ is α. Then the membership degree of fuzzy cardinality |A| is defined as,

$\mu_{|A|}(|A_\alpha|) = \alpha$, $\alpha \in \Lambda_\alpha$,Where A_α is a α-cut set and Λ_α is a level set.

In the example 1.4, If we cut fuzzy set "senior" at α=0.1, there are 5 elements in the α-cut set.

senior0.1 = {45, 55, 65, 75, 85}, |senior0.1| = 5.

In the same manner, there are 4 elements at α=0.2, there are 3 at α=0.6, there are 2 at α=1. Therefore the fuzzy cardinality of “senior” is

|senior| = {(5, 0.1), (4, 0.2), (3, 0.6), (2,1)}.

1.5 Standard Operations of Fuzzy Set:

Complement set $\bar{A}$, union A∪B, and intersection A∩B represent the standard operations of fuzzy theory and are arranged as,

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

$$\mu_{A \cup B}(x) = \text{Max}[\mu_A(x), \mu_B(x)]$$

$$\mu_{A \cap B}(x) = \text{Min}[\mu_A(x), \mu_B(x)]$$

1.5.1 Fuzzy Complement:

Complement set $\bar{A}$ of set A carries the sense of negation. Complement set may be defined by the following function C.

Complement function C:[0,1]→[0,1] is designed to map membership function $\mu_A(x)$ of fuzzy set A to [0,1] and the mapped value is written as C[$\mu_A(x)$]

i.e. $\mu_{\bar{A}}(x) = C[\mu_A(x)]$

To be a fuzzy complement function,when the axioms should be satisfied.

(Axiom C1): C(0) = 1, C(1) = 0…………………(boundary condition)

(Axiom C2): a,b ∈ [0,1] , if a< b, then C(a)≥C(b)………(monotonic nonincreasing)
Symbols a and b stand for membership value of member x in A.

For example, when $\mu_A(x)$=a, $\mu_A(y)$=b; x,y ∈ X, if $\mu_A(x) < \mu_A(y)$, C($\mu_A(x)$) ≥C($\mu_A(y)$).

C1 and C2 are fundamental requisites to be a complement function.

hese two axioms are called “axiomatic skeleton”. For particular purposes,we can insert additional requirements.

(Axiom C3) : C is a continuous function.

(Axiom C4) : C is involutive. C(C(a)) = a, ∀ a∈[0,1]

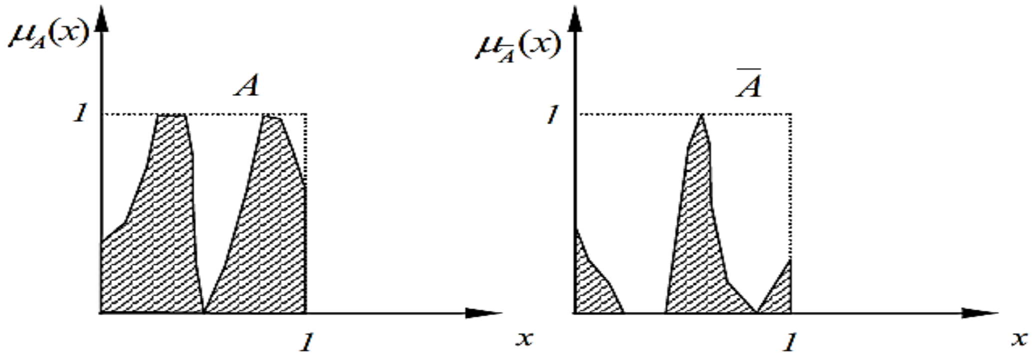


Fig 1.5 standard complement set function

Examples:

a) $$C(a) = \frac{1+\cos(\pi a)}{2}$$

(C1) C(0) = 1 and C(1) = 0

(C2) C'(a) = –½ πsin(π a) < 0 since sin(π a) > 0 for a ∈ (0,1)

(C3) Is continuous as a composition of continuous functions

(C4) not valid → counter example

$$c\left(c\left(\frac{1}{3}\right)\right) = c\left(\frac{3}{4}\right) = \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right) \neq \frac{1}{3}$$

b) $$C(a) = (1 - a^W)^{\frac{1}{W}}, \text{ for } w > 0 \quad (\text{Yager class})$$

(C1) C(0) = 1 and C(1) = 0

(C2) $(1 - a^W)^{\frac{1}{W}} \geq (1 - b^W)^{\frac{1}{W}}$

=> $(1 - a^W) \geq (1 - b^W)$

=> $a^W \leq b^W$

=> $a \leq b$

(C3) Is continuous as a composition of continuous functions

(C4)

$$c(c(a)) = c\left((1-a^w)^{\frac{1}{w}}\right) = \left(1-\left[(1-a^w)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}}$$

$$= (1-(1-a^w))^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$$

c) $C(a) = \frac{1-a}{1-\lambda a}$ **for** $\lambda > -1$ **(Sugeno class)**

(C1) C(0) = 1 and C(1) = 0

(C2)

$$c(a) \geq c(b) \Leftrightarrow \frac{1-a}{1+\lambda a} \geq \frac{1-b}{1+\lambda b} \Leftrightarrow$$

$$(1-a)(1+\lambda b) \geq (1-b)(1+\lambda a) \Leftrightarrow$$

$$b(\lambda+1) \geq a(\lambda+1) \Leftrightarrow b \geq a$$

(C3) Is continuous as a composition of continuous functions

(C4)

$$c(c(a)) = c\left(\frac{1-a}{1+\lambda a}\right) = \frac{1-\frac{1-a}{1+\lambda a}}{1+\lambda\frac{1-a}{1+\lambda a}} = \frac{a(\lambda+1)}{\lambda+1} = a$$

1.5.2 Fuzzy Union:

Union of A and B is specified by a function of the form.

U:[0,1]×[0,1]→[0,1]

This union function calculates the membership degree of union A∪B from those of A and B.

$$\mu_{A\cup B}(x) = U(\mu_A(x), \mu_B(x))$$

This union function should obey next axioms.

(Axiom U1) : U(0,0) = 0, U(0,1) = 1, U(1,0) = 1, U(1,1) = 1………(boundary condition)

(Axiom U2) : U(a,b) = U(b,a)……………….(Commutativity)

(Axiom U3) : If a≤a' and b≤b' => U(a, b) ≤ U(a', b')……….(monotonic condition)

(Axiom U4) : U(U(a, b), c) = U(a, U(b, c))……………….. (Associativity)

The above four statements are called as "axiomatic skeleton". It is often to restrict the class of fuzzy unions by adding the following axioms.

(Axiom U5): Function U is continuous.

(Axiom U6): U(a, a) = a……………….(Idempotency)

Examples of Union Function :

Yager's union function holds all axioms except U6.

$$U_w(a,b) = Min[1, (a^w + b^w)^{\frac{1}{w}}], \quad \text{where } w \in (0, \infty)$$

The shape of Yager function varies with parameter w. For instance,

w = 1 , leads to U_1(a, b) = Min[1, a+b]

w = 2 , leads to U_2(a, b) = Min[1, $\sqrt{a^2 + b^2}$]

What if w increases?

Supposing w→∞, Yager union function is transformed into the standard union function.

$$\lim_{w \to \infty} Min[1, (a^w + b^w)^{\frac{1}{w}}] = Max(a,b)$$

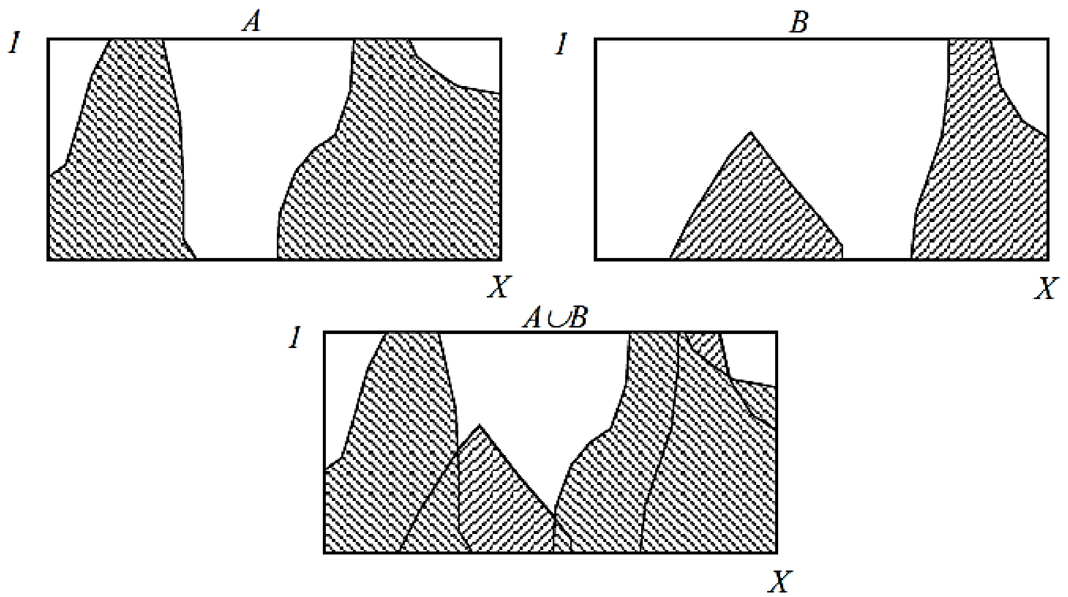


Fig 1.6 Visualization of standard union operation

1.5.3 Fuzzy Intersection :

Intersection A∩B is defined by the function I.

$$I:[0,1] \times [0,1] \rightarrow [0,1]$$

The argument of this function shows possibility for element x to be involved in both fuzzy sets A and B.

$$\mu_{A \cap B}(x) = I(\mu_A(x), \mu_B(x))$$

Intersection function holds the following axioms:

(Axiom I1) : I(1, 1) = 1, I(1, 0) = 0, I(0, 1) = 0, I(0, 0) = 0……….(boundary condition)

(Axiom I2) : I(a, b) = I(b, a)……………………..(Commutativity)

(Axiom U3) : If a≤a' and b≤b' => I(a, b) ≤ I(a', b')………..(monotonic condition)

(Axiom U4) : I(I(a, b), c) = I(a, I(b, c))……………….. (Associativity)

The above four statements are called as "axiomatic skeleton".

(Axiom U5): Function I is continuous.

(Axiom U6): I(a, a) = a……………….(Idempotency)

Examples of Intersection Function :

Yager's Intersection function holds all axioms except I6.

$$I_w(a, b) = 1 - Min[1, ((1-a)^w + (1-b)^w)^{\frac{1}{w}}], \quad \text{where } w \in (0, \infty)$$

The shape of Yager function varies with parameter w. For instance,

w = 1 , leads to I_1(a, b) = 1-Min[1, 2-a-b]

w = 2 , leads to I_2(a, b) =1- Min[1, $\sqrt{(1-a)^2 + (1-b)^2}$]

What if w increases?

Supposing w→∞, Yager union function is transformed into the standard union function.

$$\lim_{w \to \infty}(1 - Min[1, ((1-a)^w + (1-b)^w)^{\frac{1}{w}}] = Min(a, b)$$

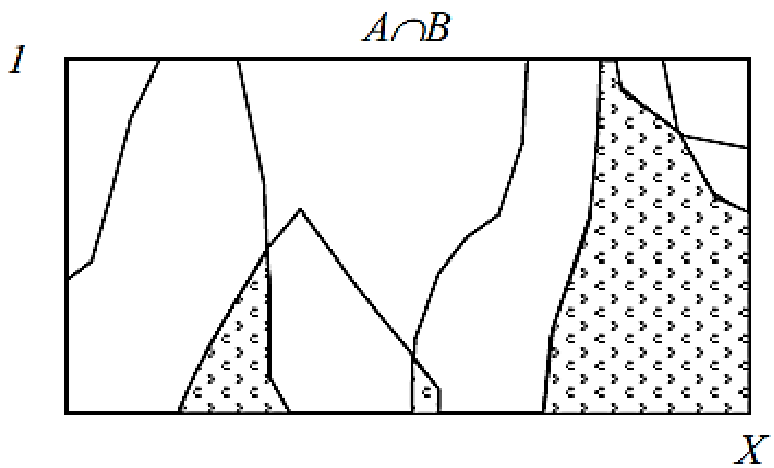


Fig 1.7 Visualization of standard fuzzy intersection set

1.6 Distance in Fuzzy Set :

(1.6.1) Hamming distance: This concept is marked as,

$$d(A,B) = \sum_{i=1, x_i \in X}^{n} \left| \mu_A(x_i) - \mu_B(x_i) \right|$$

Example: Following A and B for instance,

A = {(x1, 0.4), (x2, 0.8), (x3, 1), (x4, 0)}

B = {(x1, 0.4), (x2, 0.3), (x3, 0), (x4, 0)}

Hamming distance , d(A, B) = |0| + |0.5| + |1| + |0| = 1.5

Assuming n elements in universal set X; i.e., |X| = n, the relative Hamming distance is,

$$\delta(A,B) = \frac{d(A,B)}{n}$$

(1.6.2) Euclidean distance:

$$e(A,B) = \sqrt{\sum_{i=1}^{n} (\mu_A(x) - \mu_B(x))^2}$$

Euclidean distance between sets A and B used for the previous Hamming distance is

$$e(A,B) = \sqrt{0^2 + 0.5^2 + 1^2 + 0^2} = \sqrt{1.25} = 1.12$$

and relative Euclidean distance is $\varepsilon(A,B) = \frac{e(A,B)}{n}$.

(1.6.3) Minkowski distance :

$$d_w(A,B) = \left(\sum_{x \in X} \left| \mu_A(x) - \mu_B(x) \right|^w \right)^{\frac{1}{w}}, w \in [1, \infty)$$

Generalizing Hamming distance and Euclidean distance results in Minkowski distance. It becomes the Hamming distance for w = 1 while the Euclidean distance for w = 2.

1.7 t-norms and t-conorms:

1.7.1 Definition (t-norm):

$T : [0,1]\times[0,1]\rightarrow[0,1]$

$\forall x, y, x', y', z \in [0,1]$

i) T(x, 0) = 0, T(x, 1) = x……………..boundary condition

ii) T(x, y) = T(y, x)…………………commutativity

iii) (x ≤x', y ≤y') →T(x, y)≤T(x', y')………… monotonicity

iv) T(T(x, y), z) = T(x, T(y, z)) ………..associativity

Now we can easily recognize that the following operators hold conditions for t-norm
(1) Intersection operator
(2) Algebraic product operator
(3) Bounded product operator
(4) Drastic product operator

1.7.2 Definition (t-conorm (s-norm)):

$$\bot: [0,1]\times[0,1]\rightarrow[0,1]$$
$$\forall x, y, x', y', z \in[0,1]$$

i) ⊥ (x, 0) = x, ⊥ (x, 1) = 1……..boundary condition
ii) ⊥ (x, y) = ⊥ (y, x) ……………commutativity
iii) (x ≤x', y≤y')→ ⊥ (x, y)≤ ⊥ (x', y')………monotonicity
iv) ⊥ (⊥ (x, y), z) = ⊥ (x, ⊥ (y, z))……….associativity

examples of t-conorm operators :
(1) Union operator
(2) Algebraic sum operator
(3) Bounded sum operator
(4) Drastic sum operator
(5) Disjoint sum operator
All t-norm and t-conorm functions follow : T(a, b) ≤ Min[a, b] & ⊥(a, b) ≤Max[a, b]

1.8 Cartesian product :

Denoting $\mu_{A_1}(x), \mu_{A_2}(x), \dots, \mu_{A_n}(x)$ as membership functions of $A_1, A_2, \dots, A_n$ for $\forall\, x_1 \in A_1, x_2 \in A_2, \dots, x_n \in A_n$, then the probability for n-tuple $(x_1, x_2, \dots, x_n)$ to be involved in fuzzy set $A_1 \times A_2 \times \dots \times A_n$ is,

$$\mu_{A_1\times A_2\times\dots\times A_n}(x_1, x_2, \dots, x_n) = Min[\mu_{A_1}(x_1), \mu_{A_2}(x_2), \dots, \mu_{A_n}(x_n)]$$

1.9 Fuzzy relation:

Fuzzy relation has degree of membership whose value lies in [0, 1]

$$\mu_R: A \times B \rightarrow [0,1]$$
$$R = \{(x, y), \mu_R(x, y)) | \mu_R(x, y) \geq 0, x \in A, y \in B\}$$

1.9.1 Domain and range of fuzzy relation:

When fuzzy relation R is defined in crisp sets A and B, the domain and range of this relation are defined as

$$\mu_{dom(R)}(x) = \max_{y \in B} \mu_R(x, y)$$

$$\mu_{ran(R)}(y) = \max_{x \in A} \mu_R(x, y)$$

Set A becomes the support of dom(R) and dom(R)⊆A . Set B is the support of ran(R) and ran(R)⊆B.

1.10 Fuzzy Matrix:

Given a certain vector, if an element of this vector has its value between 0 and 1, we call this vector a fuzzy vector. Fuzzy matrix is a gathering of such vectors. Given a fuzzy matrix A=(a_{ij}) and B=(b_{ij}), we can perform operations on these fuzzy matrices

(1) Sum.................$A + B = Max(a_{ij}, b_{ij})$

(2) Max product.......... $A \bullet B = AB = \max_k \left[\min(a_{ik}, b_{kj}) \right]$

(3) Scalar product............ λA, where $0 \le \lambda \le 1$

Example: $A = \begin{pmatrix} 0.2 & 0.5 & 0.0 \\ 0.4 & 1.0 & 0.1 \\ 0.0 & 1.0 & 0.0 \end{pmatrix}$, $B = \begin{pmatrix} 1.0 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.5 \\ 0.0 & 1.0 & 0.1 \end{pmatrix}$; $A + B = \begin{pmatrix} 1.0 & 0.5 & 0.0 \\ 0.4 & 1.0 & 0.5 \\ 0.0 & 1.0 & 0.1 \end{pmatrix}$

Suppose $C = A \bullet B$ then $C_{12} = 0.1$ is calculated by applying the Max-Min operation to the values of the first row (0.2, 0.5 and 0.0) of A , and those of the second column (0.1, 0.0 and 1.0) of B.

$$\begin{array}{llll} & 0.2 & 0.5 & 0.0 \\ \text{Min}\Downarrow & & & \\ & 0.1 & 0.0 & 1.0 \\ \hline & 0.1 & 0.0 & 0.0 & \quad \text{Max}\Rightarrow\ 0.1 \end{array}$$

In the same manner $C_{13} = 0.5$

$$\therefore\ C = A \bullet B = \begin{pmatrix} 0.2 & 0.1 & 0.5 \\ 0.4 & 0.1 & 0.5 \\ 0.0 & 0.0 & 0.5 \end{pmatrix}$$

1.10.1 Fuzzy relation matrix:

If a fuzzy relation R is given in the form of fuzzy matrix, its elements represent the membership values $\mu_R(i, j)$ of this relation then , $M_R = (\mu_R(i, j))$

Example:

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

M_S	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.6	0.9	0.3

$M_{R \cup S}$	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.6	1.0	0.3

$M_{R \cap S}$	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.0	0.9	0.0

1.10.2 Composition of fuzzy relation:

Two fuzzy relations R and S are defined on sets A, B and C. That is, R⊆A×B,S⊆B×C. The composition $S \bullet R$=SR of two relations R and S is expressed by the relation from A to C, and this composition is defined by the following

$$\text{for } (x,y) \in A \times B, (y,z) \in B \times C,$$

$$\text{then } \mu_{S\bullet R}(x,z) = \max_y[\min(\mu_R(x,y), \mu_S(y,z))]$$

That is, $S \bullet R \subseteq A \times C$ and $M_{S\bullet R} = M_R \bullet M_S$

Example:

Let A={1,2,3} , B={a,b,c,d} , C={α, β, γ}; The sets A, B and C shall be the sets of events.

Consider fuzzy relations, R⊆A×B, S⊆B×C,
By the relation R, we can see the possibility of occurrence of B after A, and by S, that of C after B.

For example,

by M_R, the possibility of a∈B after 1∈A is 0.1

by M_S, the possibility of occurrence of α after a is 0.9

we cannot guess the possibility of C when A is occurred. So our main job now will be the obtaining the composition $S \bullet R$⊆A×C

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

Suppose,

$$\mu_{S\bullet R}(1,\alpha)=\max_y[\min\{\mu_R(1,y),\mu_S(y,\alpha)\}]$$

$$=\max_y[\min\{\mu_R(1,a),\mu_S(a,\alpha)\},\min\{\mu_R(1,b),\mu_S(b,\alpha)\},\min\{\mu_R(1,c),\mu_S(c,\alpha)\},\min\{\mu_R(1,d),\mu_S(d,\alpha)\}]$$

$$=\max_y[\min\{0.1,0.9\},\min\{0.2,0.2\},\min\{0.0,0.8\},\min\{1.0,0.4\}]$$

$$=\max_y[0.1,0.2,0.0,0.4]$$

$$=0.4$$

$$\mu_{S\bullet R}(3,\beta)=\max_y[\min\{\mu_R(3,y),\mu_S(y,\beta)\}]$$

$$=\max_y[\min\{\mu_R(3,a),\mu_S(a,\beta)\},\min\{\mu_R(3,b),\mu_S(b,\beta)\},\min\{\mu_R(3,c),\mu_S(c,\beta)\},\min\{\mu_R(3,d),\mu_S(d,\beta)\}]$$

$$=\max_y[\min\{0.8,0.0\},\min\{0.9,1.0\},\min\{0.1,0.0\},\min\{0.4,0.2\}]$$

$$=\max_y[0.0,0.1,0.9,0.2]$$

$$=0.9$$

In the same manner rest of the elements, ∴

S • R	α	β	γ
1	0.4	0.2	0.3
2	0.3	0.3	0.3
3	0.8	0.9	0.8

1.10.3 α-cut of Fuzzy Relation:

We can obtain α -cut relation from a fuzzy relation by taking the pairs which have membership degrees no less than α .

Assume R⊆A×B, and R_α is a α-cut relation.

$R_\alpha=\{(x,y)|\mu_R(x,y)\geq\alpha, x\in A, y\in B\}$, R_α is a crisp relation.

Example: $M_S=$

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

, Level set $\Lambda=\{0.0,0.2,0.3,0.4,0.7,0.8,0.9,1.0\}$

Now, $M_{S_{0.4}}=$

S	α	β	γ
a	1	0	0
b	0	1	1
c	1	0	1
d	1	0	0

$M_{S_{0.8}}=$

S	α	β	γ
a	1	0	0
b	0	1	1
c	1	0	0
d	0	0	0

1.11 Extension principle:

Let X be Cartesian product of universal set $X=X_1 \times X_2 \times \ldots \times X_r$ and $A_1, A_2, \ldots, A_r$ be r fuzzy sets in the universal set.
Then,

$$\mu_{A_1 \times A_2 \times \ldots \times A_r}(x_1, x_2, \ldots, x_r) = Min[\mu_{A_1}(x_1), \ldots, \mu_{A_r}(x_r)]$$

Let function f be from space X to Y, $f : X \rightarrow Y$

Then fuzzy set B in Y can be obtained by function f and fuzzy sets $A_1, A_2, \ldots, A_r$ as follows:

$$\mu_B(y) = \begin{cases} 0\ , & \text{if } f^{-1}(y) = \varnothing \\ \underset{y=f(x_1 \times x_2 \times \ldots \times x_r)}{Max} \left[Min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_r}(x_r)\} \right], & otherwise \end{cases}$$

1.11.1 Extension of fuzzy relation:

For given fuzzy set A, crisp set B and fuzzy relation R⊆A×B, there might be a mapping function expressing the fuzzy relation R. Membership function of fuzzy set B' in B is defined as follows :

For x∈A, y∈B, and B'⊆B, $\mu_{B'}(y) = \underset{y=f^{-1}(y)}{Max} \left[Min\{\mu_A(x), \mu_R(x,y)\} \right]$

Example:

Let, A1= {(-1,0.1), (0,0.4), (1,1), (2,0.5),(3,0.1)}
A2={(0,0.2), (1,0.4), (2,1),(4,0.4),(10,0.1)}

Now the mapping f, defined for the CRISP arguments $x_1 \in A, x_2 \in B$ by

$$z = f(x_1, x_2) = x_1 + \frac{1}{2} x_2\ ,$$

Then,

$$\tilde{B}\ (Fuzzy\ Set) = f(A1, A2) \ldots \ldots \ldots \ldots ?????$$

$A2 / A1$	$0^{<0.2>}$	$1^{<0.4>}$	$2^{<1>}$	$4^{<0.4>}$	$10^{<0.1>}$
$-1^{<0.1>}$	$-1^{<0.1>}$	$-0.5^{<0.1>}$	$0^{<0.1>}$	$1^{<0.1>}$	$4^{<0.1>}$
$0^{<0.4>}$	$0^{<0.2>}$	$0.5^{<0.4>}$	$1^{<0.4>}$	$2^{<0.4>}$	$5^{<0.1>}$
$1^{<1>}$	$1^{<0.2>}$	$1.5^{<0.4>}$	$2^{<1>}$	$3^{<0.4>}$	$6^{<0.1>}$
$2^{<0.5>}$	$2^{<0.2>}$	$2.5^{<0.4>}$	$3^{<0.5>}$	$4^{<0.4>}$	$7^{<0.1>}$
$3^{<0.1>}$	$3^{<0.1>}$	$3.5^{<0.1>}$	$4^{<0.1>}$	$5^{<0.1>}$	$8^{<0.1>}$

$z = x_1 + \frac{1}{2} x_2$	$Min\{\mu_{A1}(x_1), \mu_{A2}(x_2)\}$	$\max(\min s) = \mu_B(z)$
-1	0.1	0.1
-0.5	0.1	0.1
0	0.1, 0.2	0.2
0.5	0.4	0.4
1	0.1, 0.2, 0.4	0.4
1.5	0.4	0.4
2	0.4, 1, 0.2	1
2.5	0.4	0.4
3	0.4, 0.5, 0.1	0.5
3.5	0.1	0.1
4	0.1, 0.1	0.1
5	0.1, 0.1	0.1
6	0.1	0.1
7	0.1	0.1
8	0.1	0.1

$$\therefore \text{B}=\left\{\frac{-1}{0.1}, \frac{-0.5}{0.1}, \frac{0}{0.2}, \frac{0.5}{0.4}, \frac{1}{0.4}, \frac{1.5}{0.4}, \frac{2}{1}, \frac{2.5}{0.4}, \frac{3}{0.5}, \frac{3.5}{0.1}, \frac{4}{0.1}, \frac{5}{0.1}, \frac{6}{0.1}, \frac{7}{0.1}, \frac{8}{0.1}\right\}$$

1.12 Distance between fuzzy sets:

In space X, pseudo-metric distance d(A, B) between fuzzy sets A and B can be defined by extension principle. The distance d(A, B) is given as a fuzzy set.

$$\forall \delta \in \mathbb{R}^+, \mu_{d(A,B)} = \max_{\delta = d(a,b)} \left[\min\{\mu_A(a), \mu_B(b)\}\right]$$

Example:

Let , A={(1, 0.5), (2, 1), (3, 0.3)} and B={(2, 0.4), (3, 0.4), (4, 1)}

$\delta \in d(A,B)$	$a \in A$	$b \in B$	$\mu_A(a)$	$\mu_B(b)$	min	Max,$\mu(\delta)$,d(A, B)
0	2	2	1	0.4	0.4	0.4
	3	3	0.3	0.4	0.3	
1	1	2	0.5	0.4	0.4	0.4
	2	3	1	0.4	0.4	
	3	2	0.3	0.4	0.3	
	3	4	0.3	1	0.3	
2	1	3	0.5	0.4	0.4	1
	2	4	1	1	1	
3	1	4	0.5	1	0.5	1

$$\therefore d(A, B) = \left\{\frac{0}{0.4}, \frac{1}{0.4}, \frac{2}{1}, \frac{3}{1}\right\}$$

1.13 FUZZY NUMBER:

When interval is defined on real number ℝ, this interval is said to be a subset of ℝ. For instance, if interval is denoted as A = [a1, a3]; a1, a3 ∈ ℝ,a1 < a3, we may regard this as one kind of sets. Expressing the interval as membership function is shown in the following

$$\mu_A(x) = \begin{cases} 0, \text{ if } x < a1 \\ 1, \text{ if } a1 \le x \le a3 \\ 0, \text{ if } x > a3 \end{cases}$$

If a1 = a3, this interval indicates a point i.e , [a1, a1] = a1

Definition:

If a fuzzy set satisfy the following conditions :

- convex fuzzy set
- normalized fuzzy set
- it's membership function is piecewise continuous.
- It is defined in the real number
- It should have bounded support

The convex condition is that the line by α-cut is continuous and α –cut interval satisfies the following relation

$$A_\alpha = [a_1^\alpha, a_2^\alpha]$$
$$\alpha < \beta \Rightarrow (a_1^\alpha < a_1^\beta, a_2^\beta < a_2^\alpha)$$

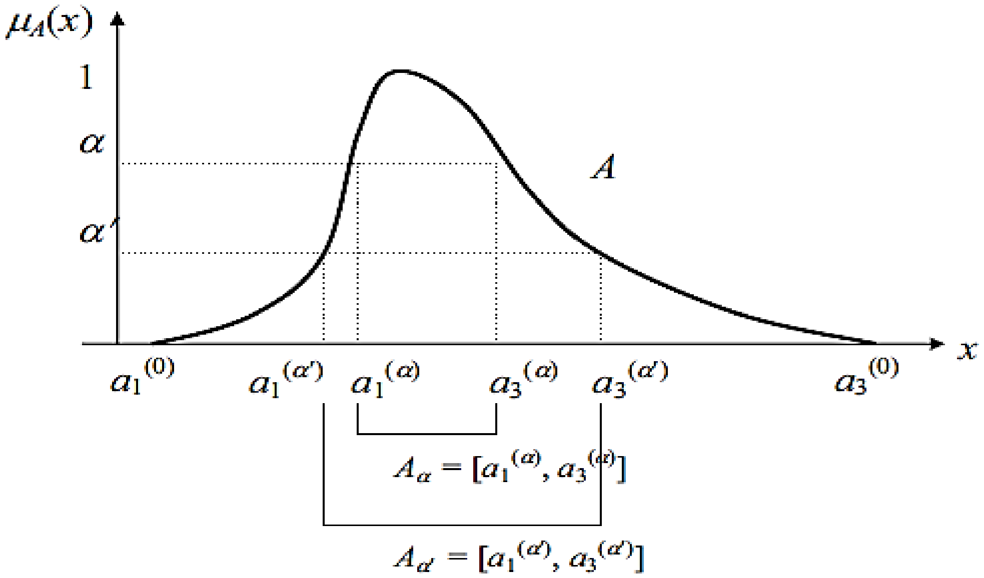


Fig 1.8 ***α***-cut of fuzzy number $(\alpha' < \alpha) \Rightarrow A_\alpha \subset A_{\alpha'}$

1.13.1 Arithmetic Operations on Fuzzy Numbers:

In Classical interval analysis, ∃ 4 operations, namely +, - , · , /

Let denote, $*$ any of the 4 operations then $[a,b]*[d,e]=\{f*g \mid a\le f\le b, d\le g\le e\}$ except $[a,b]/[d,e]$ when $0\in[d,e]$.

α-cut pproach:

> Let A and B are two fuzzy no's & A_α and B_α be α-cuts of A,B respectively, $\alpha\in[0,1]$
> then, ${}^\alpha(A*B)={}^\alpha A*{}^\alpha B$ and $A*B=\bigcup_\alpha {}^\alpha(A*B)$

1.13.2 Type of Fuzzy Arithmetic :

$\begin{cases}\text{Interval arithmetic}\\ \text{Extension principle}\end{cases}$

- *Interval arithmetic method :*

Definitions of operations of Intervals:

- ✓ $[a,b]+[d,e]=[a+b,b+e]$
- ✓ $[a,b]-[d,e]=[a-b,b-e]$
- ✓ $[a,b]\bullet[d,e]=[\min(ad,ae,bd,be),\max(ad,ae,bd,be)]$
- ✓ $[a,b]/[d,e]=[a,b]\bullet[1/e,1/d]=[\min(\frac{a}{d},\frac{a}{e},\frac{b}{d},\frac{b}{e}),\max(\frac{a}{d},\frac{a}{e},\frac{b}{d},\frac{b}{e})]$, where $0\notin[d,e]$

●Examples :

1. [2,5]+[1,3]=[3,8]
2. [2,5]-[1,3]=[-1,4]
3. [3,4]·[2,2]=[6,8]
4. [4,10]/[1,2]=[2,10]

- *Extension principle method* : $\mu_{A*B}(z)=\sup_{z=x*y}[\min\{\mu_A(x),\mu_B(y)\}]$

1.13.3 Operation of Fuzzy Number:

Previous operations of interval are also applicable to fuzzy number. Since outcome of fuzzy number (fuzzy set) is in the shape of fuzzy set, the result is expressed in membership function.

$$\forall x,y,z\in\mathbb{R}$$

- ✓ For Addition(+)….. $\mu_{A(+)B}(z)=\max_{z=x+y}[\min\{\mu_A(x),\mu_B(y)\}]$
- ✓ For Subtraction(-)…. $\mu_{A(-)B}(z)=\max_{z=x-y}[\min\{\mu_A(x),\mu_B(y)\}]$
- ✓ For multiplication(·)…. $\mu_{A(\cdot)B}(z)=\max_{z=x\cdot y}[\min\{\mu_A(x),\mu_B(y)\}]$
- ✓ For division(/)…. $\mu_{A(/)B}(z)=\max_{z=x/y}[\min\{\mu_A(x),\mu_B(y)\}]$

- ✓ For maximum(∨)…. $\mu_{A(\vee)B}(z) = \max\limits_{z=x\vee y}[\min\{\mu_A(x), \mu_B(y)\}]$
- ✓ For minimum(∧)…. $\mu_{A(\wedge)B}(z) = \max\limits_{z=x\wedge y}[\min\{\mu_A(x), \mu_B(y)\}]$

Examples:

Let,

A= {(2,1),(3,0.5)} ; B = {(3,1),(4,0.5)}

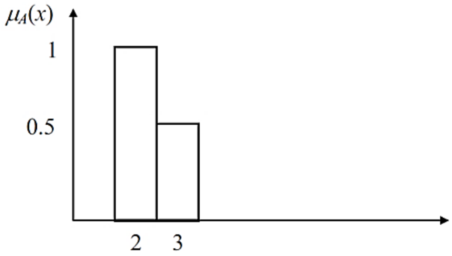


Fig 1.9(a) Fuzzy set A

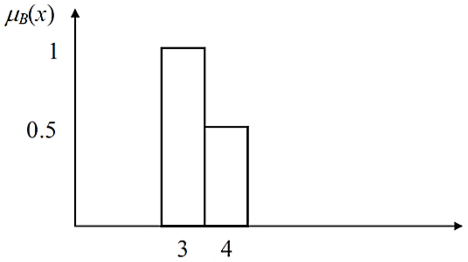


Fig 1.9(b) Fuzzy set B

$A(+)B =$

z	x	y	$\mu_A(x)$	$\mu_B(y)$	min	max(min s), $\mu_{A(+)B}(z)$
5	2	3	1	1	1	1
6	3	3	0.5	1	0.5	0.5
	2	4	1	0.5	0.5	
7	3	4	0.5	0.5	0.5	0.5

$A(-)B =$

z	x	y	$\mu_A(x)$	$\mu_B(y)$	min	max(min s), $\mu_{A(-)B}(z)$
−2	2	4	1	0.5	0.5	0.5
−1	2	3	1	1	1	1
	3	4	0.5	0.5	0.5	
0	3	3	0.5	1	0.5	0.5

1.13.4 <u>Types of Fuzzy Nomber:</u>

1) Triangular fuzzy no
2) Trapezoidal fuzzy no
3) Gaussian fuzzy no
4) Quasi- Gaussian fuzzy no
5) Quadratic fuzzy no
6) Exponential fuzzy no
7) Quasi- Exponential fuzzy no
8) Fuzzy singleton etc.

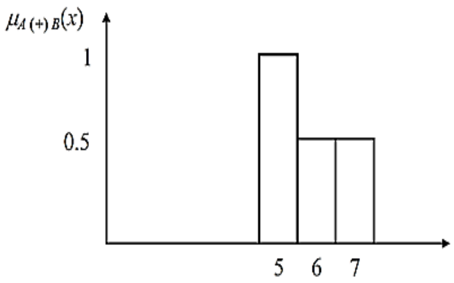
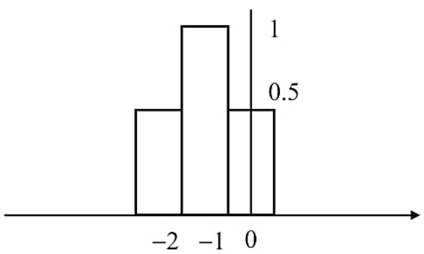


Fig 1.10



Here we Discuss only about Triangular and Trapezoidal no's….

1.13.4.1 *Triangular fuzzy no:*

It is a fuzzy number represented with three points as follows : $A=(a_1,a_2,a_3)$
This representation is interpreted as membership functions

$$\mu_A(x)=\begin{cases} 0 & ,x<a_1 \\ \dfrac{x-a_1}{a_2-a_3} & ,a_1\le x\le a_2 \\ \dfrac{a_3-x}{a_3-a_2} & ,a_2\le x\le a_3 \\ 0 & ,a_3<x \end{cases}$$

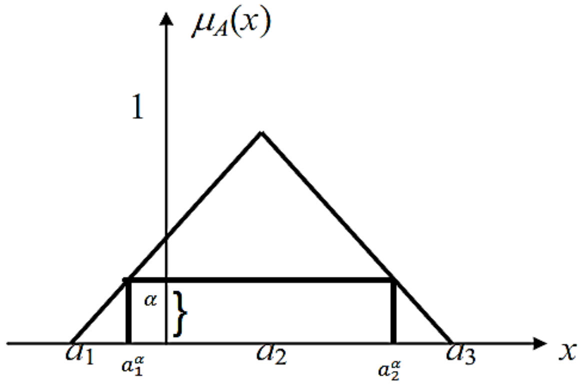


Fig 1.11 α-cut of triangular fuzzy number $A=[a_1,a_2,a_3]$

Now if you get crisp interval by α-cut operation.

Then interval A_α shall be obtained as $[a_1^\alpha,a_2^\alpha],\forall\alpha\in[0,1]$
From the above picture ,

$$\frac{a_1^\alpha-a_1}{a_2-a_1}=\frac{a_3-a_2^\alpha}{a_3-a_2}=\alpha$$

$$\Rightarrow a_1^\alpha=(a_2-a_1)\alpha-a_1 \;\&\; a_2^\alpha=a_3-(a_3-a_2)\alpha$$

$$\therefore A_\alpha=[(a_2-a_1)\alpha-a_1,a_3-(a_3-a_2)\alpha]$$

Example:

In the case of the triangular fuzzy number A = (-5, -1, 1), the membership function value will be,

$$\mu_A(x)=\begin{cases} 0 & ,x<-5 \\ \dfrac{x+5}{4} & ,-5\le x\le -1 \\ \dfrac{1-x}{2} & ,-1\le x\le 1 \\ 0 & ,1<x \end{cases}$$

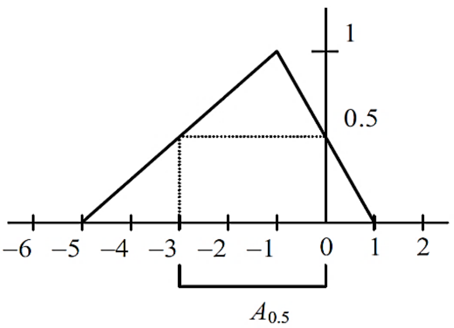


Fig 1.12 α=0.5 cut of triangular fuzzy number

A = (-5,-1, 1)

here, $A_\alpha=[(-1+5)\alpha+5,1-(1+1)\alpha]$

$=[4\alpha+5,1-2\alpha]$

Operation of Triangular Fuzzy Number:

- ✓ The results from addition or subtraction between triangular fuzzy numbers result also triangular fuzzy numbers.
- ✓ The results from multiplication or division are not triangular fuzzy numbers.
- ✓ Max or min operation does not give triangular fuzzy number.

Here we have to deal the operations with an example of their membership functions.
Let ,

$$\mu_A(x)=\begin{cases}0 & ,x<-1\\ \dfrac{x+1}{2} & ,-1\le x\le 1\\ \dfrac{3-x}{2} & ,1\le x\le 3\\ 0 & ,3<x\end{cases} \quad \text{and} \quad \mu_B(x)=\begin{cases}0 & ,x\le 1\\ \dfrac{x-1}{2} & ,1< x\le 3\\ \dfrac{5-x}{2} & ,3\le x\le 5\\ 0 & ,5<x\end{cases}$$

$$\therefore\ A=[-1,1,3];\ B=[1,3,5]$$

$$\text{and } A^{\alpha}=[2\alpha-1,3-2\alpha];\ B^{\alpha}=[2\alpha+1,5-2\alpha]$$

$$(A+B)^{\alpha}=[4\alpha,8-4\alpha]$$

$$(A-B)^{\alpha}=[4\alpha-6,2-4\alpha]$$

$$(A\cdot B)^{\alpha}=\begin{cases}[-4\alpha^2+12\alpha-5,4\alpha^2-16\alpha+15],\ for\ \ \alpha\in[0,0.5]\\ [4\alpha^2-1,4\alpha^2-16\alpha+15],\ for\ \ \alpha\in[0.5,1]\end{cases}$$

$$(A/B)^{\alpha}=\begin{cases}[(2\alpha-1)/(2\alpha+1),(3-2\alpha)/(2\alpha+1)],\ for\ \ \alpha\in[0,0.5]\\ [(2\alpha-1)/(5-2\alpha),(3-2\alpha)/(2\alpha+1)],\ for\ \ \alpha\in[0.5,1]\end{cases}$$

$$\because \text{ we know that } A*B=\bigcup_{\alpha}(A*B)^{\alpha},\forall\alpha\in[0,1]$$

$$A+B=[0,4,8]\ ;\ A-B=[-6,-2,2]\ ;\ \underbrace{A\cdot B=[-5,3,15]}_{\text{but not triangular no}}\ ;\ \underbrace{A/B=[-1,\tfrac{1}{3},3]}_{\text{not triangular no}}$$

$$\mu_{(A+B)}(x)=\begin{cases}0,\ \text{for } x\le 0\\ x/4,\ \text{for } 0<x\le 4\\ (8-x)/4,\ \text{for } 4<x\le 8\\ 0,\ \text{for } 8<x\end{cases}$$

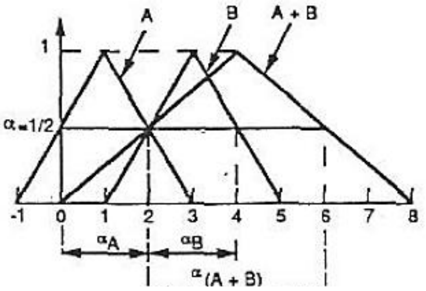


Fig 1.13 A+B

$$\mu_{(A-B)}(x) = \begin{cases} 0 \text{ , for -6} \le \text{x and x>2} \\ (x+6)/4 \text{ , for -6<x} \le \text{-2} \\ (2-x)/4 \text{ , for -2<x} \le 2 \end{cases}$$

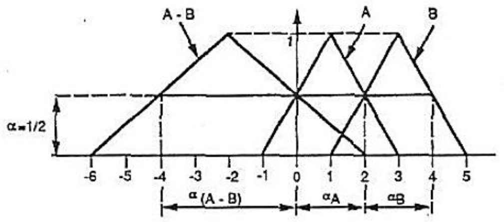


Fig 1.14 A-B

$$\mu_{(A\cdot B)}(x)(x) = \begin{cases} 0 \text{ , for -5<x and 15} \le \text{x} \\ (3-(4-x)^{1/2})/2 \text{ , for -5<x} \le 0 \\ (1+x)^{1/2}/2 \text{ , for 0<x} \le 3 \\ [4-(1+x)^{1/2}]/2 \text{ , for 3} \le \text{x<15} \end{cases}$$

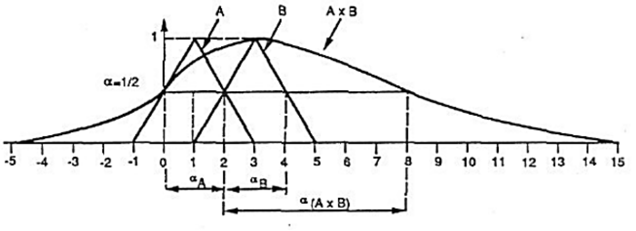


Fig 1.15 A.B

$$\mu_{(A/B)}(x) = \begin{cases} 0 \text{ , for -1<x and 3} \le \text{x} \\ (x+1)/(2-2x) \text{ , for -1} \le \text{x<0} \\ (1+5x)/(2x+2) \text{ , for 0<x} \le 1/3 \\ (3-x)/(2x+2) \text{ , for 1/3} \le \text{x<3} \end{cases}$$

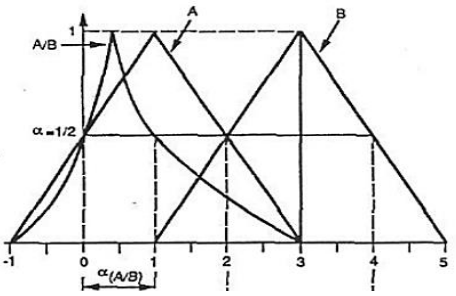


Fig 1.16 A/B

1.13.4.2 Trapezoidal fuzzy number:

We can define trapezoidal fuzzy number A as A = (a_1, a_2, a_3, a_4)

$$\mu_A(x) = \begin{cases} 0 & , x < a_1 \\ \dfrac{x-a_1}{a_2-a_1} & , a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{a_3-x}{a_4-a_3} & , a_3 \le x \le a_4 \\ 0 & , a_4 < x \end{cases}$$

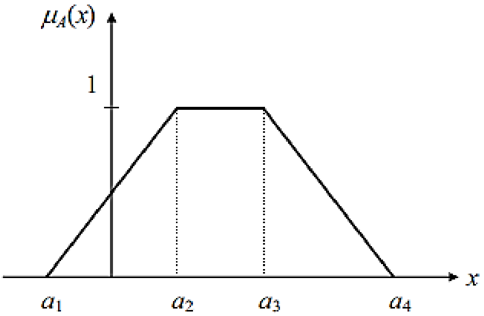


Fig 1.17 trapezoidal fuzzy number A = (a_1, a_2, a_3, a_4)

Operations of Trapezoidal Fuzzy Number similar to the Triangular Fuzzy Number.

Example:

Let two trapezoidal fuzzy numbers as A = (1, 5, 6, 9);B = (2, 3, 5, 8)

$A_\alpha = [4\alpha + 1, -3\alpha + 9]$

$B_\alpha = [\alpha + 2, -3\alpha + 8]$

$\because \forall \alpha \in$ [0, 1], each element for each interval is positive, multiplication between α-cut intervals will be

$A_\alpha \cdot B_\alpha = [4\alpha^2 + 9\alpha + 2, 9\alpha^2 - 51\alpha + 72]$

if $\alpha = 0, A_0 \cdot B_0 = [2, 72]$

if $\alpha = 1, A_1 \cdot B_1 = [15, 30]$

$\therefore A \cdot B \cong [2, 15, 30, 72]$

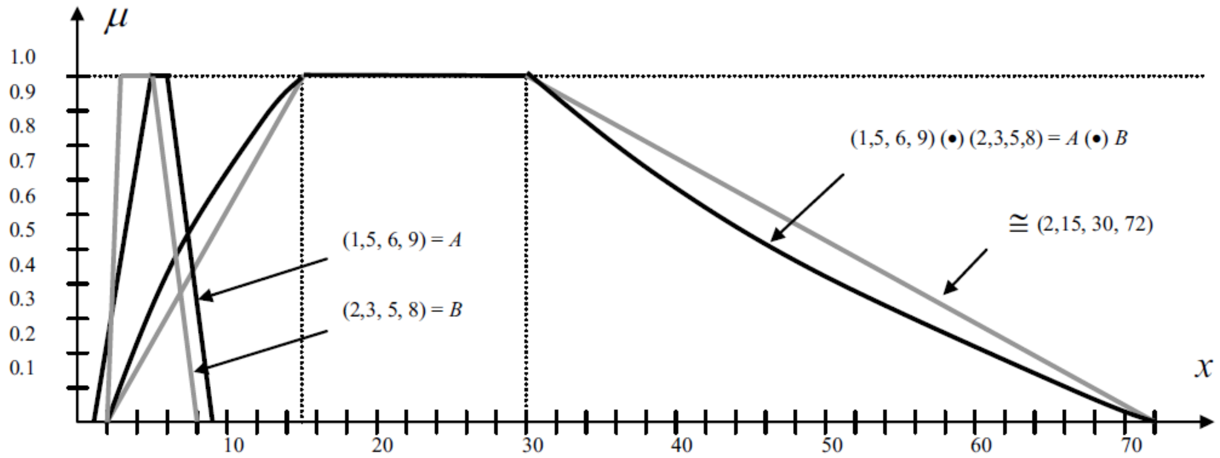


Fig 1.18 Multiplication of trapezoidal fuzzy number

1.14 <u>FUZZY FUNCTION:</u>

Kinds of Fuzzy Function:

1) Crisp function with fuzzy constraint.
2) Crisp function which propagates the fuzziness of independent variable to dependent variable.
3) Function that is itself fuzzy. This fuzzifying function blurs the image of a crisp independent variable.

1.14.1 Crisp function with fuzzy constraint:

Let X and Y be crisp sets,and f be a crisp function. A and B are fuzzy sets defined on universal sets X and Y respectively. Then the function satisfying the condition $\mu_A(x) \le \mu_B(f(x))$ is called a function with constraints on fuzzy domain A and fuzzy range B.

Example:

We shall investigate a function with the following statement.

" A competent salesman gets higher income"

Let X and Y be sets of salesmen and of monthly income [0,∞] respectively. And A and B are fuzzy sets of "competent salesmen" and "high income".
In this case, the functions $f: A \to B$ satisfy $\forall x \in A$ and $y = f(x) \in B$, $\mu_A(x) \le \mu_B(f(x))$

1.14.2 Propagation of Fuzziness by Crisp Function:

Fuzzy extension function propagates the ambiguity of independent variables to dependent variables when f is a crisp function from X to Y, the fuzzy extension function f defines the image $f(\tilde{X})$ of fuzzy set $\tilde{X}$. That is, the extension principle is applied

$$\mu_{f(\tilde{x})}(y) = \begin{cases} 0\ , & \text{if } f^{-1}(y) = \varnothing \\ \underset{f^{-1}(y)}{Max}\ \mu_{\tilde{x}}(x), & otherwise \end{cases}$$

Here we use the sign ~ for the emphasis of fuzzy variable.
Example:

There is a crisp function, $f(x) = 3\tilde{x} + 1$
Where its domain is A = {(0,0.9), (1,0.8),(2,0.7),(3, 0.6),(4,0.5)} and its range is B = [0,20].

The independent variables have ambiguity and the fuzziness is propagated to the crisp set B .Then, we can obtain a fuzzy set B' in B

B'= {(1,0.9), (4,0.8), (7,0.7), (10,0.6), (13,0.5)}

1.14.3 Fuzzifying Function of Crisp Variable:

Fuzzifying function from X to Y is the mapping of X in fuzzy power set $\tilde{P}(Y)$.

$$f: X \to \tilde{P}(Y)$$

Example:

Consider two crisp sets A = {2, 3, 4} and B = {2, 3, 4, 6, 8, 9,12}
A fuzzifying function $\tilde{f}$ maps the elements in A to power set $\tilde{P}(B)$ in the following manner

$$\tilde{f}(2) = B1, \tilde{f}(3) = B2, \tilde{f}(4) = B3$$

Where $\tilde{P}(B)$={B1,B2,B3}
B1 = {(2, 0.5), (4, 1), (6. 0.5)} B2 = {(3. 0.5), (6,1), (9, 0.5), B3 ={(4,0.5), (8, 1), (12, 0.5)}

1.15 Fuzzy bunch of functions:

Fuzzy bunch of crisp functions from X to Y is defined with fuzzy set of crisp function $f_i (i = 1,...,n)$ and it is denoted as

$$\tilde{f} = \{(f_i, \mu_{\tilde{f}}(f_i)) | f_i: X \to Y\}$$

Example:
Let X = {1, 2, 3} and $\tilde{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.5)\}$

$$f_1(x) = x, f_2(x) = x^2, f_3(x) = -x + 1$$

X	$f_1^{0.4}(x)=x$	$f^{0.7}{}_2(x)=x^2$	$f^{0.5}{}_3(x)=-x+1$	$\tilde{f}$
1	$1^{0.4}$	$1^{0.7}$	$0^{0.5}$	$\tilde{f}(1)=\{1^{0.7},0^{0.5}\}$
2	$2^{0.4}$	$4^{0.7}$	$-1^{0.5}$	$\tilde{f}(2)=\{-1^{0.5},2^{0.4},4^{0.7}\}$
3	$3^{0.4}$	$9^{0.7}$	$-2^{0.5}$	$\tilde{f}(3)=\{-2^{0.5},3^{0.4},9^{0.7}\}$

1.16 Fuzzy Integration:

1.16.1) Integration of fuzzifying function in crisp interval:

In non-fuzzy interval [a, b],let the fuzzifying function have fuzzy value $\tilde{f}(x)$ for x∈[a, b].Integration $\tilde{I}(a,b)$ of the fuzzifying function in [a, b] is defined as

$$\tilde{I}(a,b) = \left[\left\{\left(\int_a^b f_\alpha^+(x)dx + \int_a^b f_\alpha^-(x)dx\right), \alpha\right\} | \alpha \in [0,1]\right]$$

Here f_α^+ and f_α^- are α-cut functions of $\tilde{f}(x)$. Note that the plus sign(+) in the above formula is to express enumeration in fuzzy set but not addition. Therefore, the total integration is obtained by aggregating Integrations of each α-cut function.

Example:

Let $\tilde{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.4)\}$

X=[1,2], $f_1(x) = x, f_2(x) = x^2, f_3(x) = x + 1$

Integration at α=0.7

$$f = f_2(x) = x^2$$

$$I_{0.7}(1,2) = \int_1^2 x^2 dx = \frac{7}{3}$$

$$\tilde{I}_\alpha(1,2)=\{(7/3,0.7)\}$$

Integration at α=0.4

There are two functions $f^+ = f_1(x) = x$ and $f^- = f_3(x) = x + 1$

$$I_\alpha^+(1,2) = \int_1^2 x dx = \frac{3}{2}$$

$$I_\alpha^-(1,2) = \int_1^2 (x+1) dx = \frac{5}{2}$$

$$\tilde{I}_{0.4}(1,2) = \{\left(\frac{3}{2}, 0.4\right), \left(\frac{5}{2}, 0.4\right)\}$$

$$\therefore \tilde{I}(a,b) = \left\{\left(\frac{7}{3}, 0.7\right), \left(\frac{3}{2}, 0.4\right), \left(\frac{5}{2}, 0.4\right)\right\}$$

1.16.2) Integration crisp function in fuzzy interval:

Integration I(A, B) of non-fuzzy function f in fuzzy interval [A, B] is defined as,

$$\mu_{I(A,B)}(z) = \max_{x,y}\left[\min\{\mu_A(x), \mu_B(y)\}\right], \text{where } z = \int_a^b f(u)du$$

Example:

Let A = {(4,0.8), (5,1), (6,0.4)};B = {(6,0.7), (7, 1), (8,0.2)}

$$f(x) = 2, x \in [4,8]$$

$$\tilde{I}(A,B) = \int_A^B f(x)dx = \int_A^B 2dx$$

$[a,b]$	$\int_a^b 2dx$	$\min[\mu_A(a), \mu_B(b)]$
[4,6]	4	0.7
[4,7]	6	0.8
[4,8]	8	0.2
[5,6]	2	0.7
[5,7]	4	1.0
[5,8]	6	0.2
[6,6]	0	0.4
[6,7]	2	0.4
[6,8]	4	0.2

$$\therefore \tilde{I}(A,B) = \{(0,0.4),(2,0.7),(4,1),(6,0.8),(8,0.2)\}$$

1.17 **<u>Fuzzy Differentiation:</u>**

1.17.1 Differentiation of crisp function on fuzzy points:

By the extension principle, differentiation $f'(A,B)$ of non-fuzzy function f at fuzzy point or fuzzy set A is defined as

$$\mu_{f'(A)}(y) = \max_{f(x)=y} \mu_A(x)$$

Example:

Let $A = \{(-1,0.4),(0,1),(1,0.6)\}$ and $f(x) = x^3$

$f'(x) = 3x^2$, then $f'(A) = \{(3,0.4),(0,1),(3,0.6)\} = \{(0,1),(3,0.6)\}$

1.17.2 Differentiation of fuzzifying function in crisp interval:

Similar as integration of fuzzifying function in the crisp interval.

Example:

Let $\tilde{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.4)\}$, where $f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = x^3 + 1$

$f_1'(x) = 1,\ f_2'(x) = 2x,\ f_3'(x) = 3x^2$

$f_1'(0.5) = 1$, where $\alpha = 0.4$

$f_2'(0.5) = 1$, where $\alpha = 0.7$

$f_3'(0.5) = 0.75$, where $\alpha = 0.4$

$$\therefore \left(\frac{\widetilde{df}}{dx}\right)_{0.5} = \{(1, 0.4), (1, 0.7), (0.75, 0.4)\}$$

$$= \{(1, 0.7), (0.75, 0.4)\}$$

Perspectives of Fuzzy Initial Value Problems

In the modeling of real world phenomena, often some or most of the pertinent information may be uncertain. For example, the precise initial state may not beknown or information about various parameters required as a part of the model may be imprecise.Many times, the nature of the uncertainty involved may not be statistical. In such situations involving uncertainties, Fuzzy deferential equations(FDEs) are a natural way to model dynamical systems. Here, we are interested in issues concerning Fuzzy Initial Value Problems (FIVP).

Consider the initial value problem (IVP for short) for the fuzzy differential equation

$$u' = f(t,u), u(t_0) = u_0 \; ; \; t_0 \geq 0 \quad (1)$$

Where $f: J \times E^n \to E^n; J = [t_0, t_0 + a], a > 0$ and (E^n, d) is a complete fuzzy metric space.

Let us first note that a mapping $u: J \to E^n$ is a solution of the IVP (1) if and only if it is continuous and satisfies the integral equation

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds, \quad t \in J \quad (2)$$

We also observe that if u(t) satisfies (1) then $dia[u(t)]^{\alpha} \geq dia[u_0]^{\alpha}, \; \alpha \in [0,1]$ where diam means the diameter of the set involved.

2.1 Lipschitz condition for possesses a unique solution:

$$d[f(t,u), f(t,v)] \leq kd[u,v] \quad (3)$$

where $u, v \in E^n$. Then (1) has unique solution u(t) on J.

Proof: Reference

Example:

Let $A, B: J \to E^1$ be continuous. Define $f: J \times E \to E$ by $f(t,u) = A(t)u + B(t)$, where the multiplication in E' is given by Zadeh's extension principle.

Let $[A(t)]^{\alpha} = [a_1^{\alpha}, a_2^{\alpha}]$ and $[x]^{\alpha} = [x_1^{\alpha}, x_2^{\alpha}]$, Then a straight forward computation shows that f (t, u) satisfies the assumptions of Lipschitz condition then (1) has unique solution on J.

2.2 Existence of IVP:

Assume that $f: J \times E^n \to E^n$ and

$$d[f(t,u),\tilde{0}] \leq M, t \in J, u \in E^n,$$

where $\tilde{0} \in E^n$ is defined as $\tilde{0}(x) = 1 \; if \; x = 0, and \; \tilde{0}(x) = 0 \; if \; x \neq 0.$

Then IVP (1) has a solution u(t) on J.

Global Existence of IVP:

Assume that $f: \mathbb{R}_+ \times E^n \to E^n$,

$$d[f(t,u),\tilde{0}] \leq g(t, d[u,\tilde{0}]), \qquad where \; (t,u) \in \mathbb{R}_+ \times E^n$$

Where : $\mathbb{R}_+^2 \to \mathbb{R}_+$, $g(t,w)$ is non-decreasing in w for each $t \in \mathbb{R}_+$ and the max solution $r(t,t_0,w_0)$ of (1) exist on $[t_0,\infty)$.

2.3 Approximate solution:

A function $v(t) = v(t,t_0,v_0,\varepsilon), \varepsilon > 0$ is said to be an ε - approximate solution of (1) if $v: \mathbb{R}_+ \to E^n$, $v(t_0,t_0,v_0,\varepsilon) = v_0$ and $d[v'(t), f(t,v(t))] \leq \varepsilon, t \geq t_0$.

In case , $\varepsilon = 0, v(t)$ is a solution of (1).

2.4 Stability Criteria:

Before we proceed further to investigate stability results of fuzzy differential equations, let us note the following fact. The solutions of fuzzy differential equations have, in general, the property that $dia[x(t)]^\alpha$ is non-decreasing as time increases. Hence the formulation we have been working with is not suitable to reflect the rich behavior of solutions of ordinary differential equations.

Example:

Let $a \in E$ have level set $[a]^\alpha = [a_1^\alpha, a_2^\alpha]$ for $\alpha \in I = [0,1]$ and suppose that a solution $x: [0,T] \to E$ of the fuzzy differential equation

$$\frac{dx}{dt} = ax, \text{on } E \qquad \text{(e\_1)}$$

has level set $[x(t)]^\alpha = [x_1^\alpha(t), x_2^\alpha(t)]$ for $\alpha \in I$ and $t \in [0,T]$

Now the Hukuhara derivative $\frac{d}{dt}x(t)$ also has level sets $[\frac{d}{dt}x(t)]^\alpha = [\frac{d}{dt}x_1^\alpha(t), \frac{d}{dt}x_2^\alpha(t)]$

By the extension principle ,the fuzzy set $f(x(t)) = ax(t)$ has level sets

$$[ax(t)]^\alpha = [\min\{a_1^\alpha x_1^\alpha, a_1^\alpha x_2^\alpha, a_2^\alpha x_1^\alpha, a_2^\alpha x_2^\alpha\}, \max\{a_1^\alpha x_1^\alpha, a_1^\alpha x_2^\alpha, a_2^\alpha x_1^\alpha, a_2^\alpha x_2^\alpha\}]$$

thus the fuzzy differential equation (e_1) is equivalent to the coupled system of differential equations

$$\frac{d}{dt}x_1^\alpha(t) = \min\{a_1^\alpha x_1^\alpha, a_1^\alpha x_2^\alpha, a_2^\alpha x_1^\alpha, a_2^\alpha x_2^\alpha\}$$
$$\frac{d}{dt}x_2^\alpha(t) = \max\{a_1^\alpha x_1^\alpha, a_1^\alpha x_2^\alpha, a_2^\alpha x_1^\alpha, a_2^\alpha x_2^\alpha\} \quad \text{(e\_2)}$$

for particular $a = -1 \in E$

then (e_1) becomes , $\frac{dx}{dt} = -x$, on E

and (e_2) becomes ,

$$\frac{d}{dt}x_1^\alpha(t) = -x_2^\alpha \text{ and } \frac{d}{dt}x_1^\alpha(t) = -x_1^\alpha \quad \text{(e\_3)}$$

Let us consider to an IV $x_0 \in E$ with $[x_0]^\alpha = [x_{01}^\alpha, x_{02}^\alpha]$ for $\alpha \in I$.

$$\frac{d^2}{dt^2}x_1^\alpha = -\frac{d}{dt}x_2^\alpha = x_1^\alpha$$

$\therefore$ A.E is, $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$\therefore x_1^\alpha = c_1 e^t + c_2 e^{-t}$,then $x_{01}^\alpha = c_1 + c_2$

$$\frac{d}{dt}x_1^\alpha = -x_2^\alpha = c_1 e^t - c_2 e^{-t}$$

$\therefore x_2^\alpha = -c_1 e^t + c_2 e^{-t}$,then $x_{02}^\alpha = -c_1 + c_2$

$\therefore c_1 = \frac{1}{2}\left(x_{01}^\alpha - x_{02}^\alpha\right)$ and $c_2 = \frac{1}{2}\left(x_{01}^\alpha + x_{02}^\alpha\right)$

$\therefore x_1^\alpha = \frac{1}{2}\left(x_{01}^\alpha - x_{02}^\alpha\right)e^t + \frac{1}{2}\left(x_{01}^\alpha + x_{02}^\alpha\right)e^{-t}$ and $x_2^\alpha = -\frac{1}{2}\left(x_{01}^\alpha - x_{02}^\alpha\right)e^t + \frac{1}{2}\left(x_{01}^\alpha + x_{02}^\alpha\right)e^{-t}$

$when\ [x_0]^\alpha = [\alpha - 1, 1 - \alpha],\ for\ \alpha \in I$

$\therefore x_1^\alpha = \frac{1}{2}\left((\alpha - 1) - (1 - \alpha)\right)e^t + \frac{1}{2}\left((\alpha - 1) + (1 - \alpha)\right)e^{-t} = (\alpha - 1)e^t$

and $x_2^\alpha = -\frac{1}{2}\left((\alpha - 1) - (1 - \alpha)\right)e^t + \frac{1}{2}\left((\alpha - 1) + (1 - \alpha)\right)e^{-t} = -(\alpha - 1)e^t$

$[x(t)]^\alpha = [(\alpha - 1)e^t, -(\alpha - 1)e^t] = (1 - \alpha)e^t[-1, 1]$

$\forall \alpha \in I$ and $t \geq 0$, $dia[x(t)]^\alpha = 2(1 - \alpha)e^t$

Hence the solution becoms fuzzier as time increases.

2.5 Definition(H-difference):

Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$; then z is called the H-difference of x and y and it is denoted by $x \ominus y$.

Here $\ominus$ sign stands always for H-difference and let us remark that $x \ominus y \neq x + (-1)y$. Usually we denote $x + (-)y$ by $x - y$; while $x \ominus y$ stands for the H-difference.

2.6 Solutions under strongly generalized differentiability:

Let $f: (a, b) \to E$ and $x_0 \in [a, b]$.We say that f is strongly generalized differentiable at x_0 ,if there exit an element $f'(x_0) \in E$,such that

(i) $\forall\, h > 0$ sufficiently small ,$\exists\, f(x_0 + h) \ominus f(x_0),\ f(x_0) \ominus f(x_0 - h)$ and the limits (in (E, d))

$$\lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

or

(ii) $\forall\, h > 0$ sufficiently small ,$\exists\, f(x_0) \ominus f(x_0 + h),\ f(x_0 - h) \ominus f(x_0)$ and the limits (in (E, d))

$$\lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or

(iii) $\forall\, h > 0$ sufficiently small ,$\exists\, f(x_0 + h) \ominus f(x_0),\ f(x_0 - h) \ominus f(x_0)$ and the limits (in (E, d))

$$\lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or

(iv) $\forall\, h > 0$ sufficiently small ,$\exists\, f(x_0) \ominus f(x_0 + h),\ f(x_0 - h) \ominus f(x_0)$ and the limits (in (E, d))

$$\lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 + h)}{h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 - h)}{-h} = f'(x_0)$$

A function that is strongly generalized differentiable as in cases (i) and (ii) , will be referred as (i)-differentiable or as (ii) differentiable, respectively.

As for cases (iii) and (iv), a function may be differentiable as in (iii) or (iv) only on a discrete set of points (where differentiability switches between cases (i) and (ii)).

Theorem: If $f(t) = \{x(t), y(t), z(t)\}$ is triangular number valued function, then

a) If u is (i)-differentiable (Hukuhara differentiable) then $f' = (x', y', z')$.

b) If u is (ii)-differentiable (Hukuhara differentiable) then $f' = (z', y', x')$.

Proof:

The proof of b) is as follows. Let us suppose that the H-difference exists. Then, by direct computation we get

$$\lim_{h\to 0}\frac{f(t)\ominus f(t+h)}{-h} = \frac{\{x(t)-x(t+h), y(t)-y(t+h), z(t)-z(t+h)\}}{-h}$$

$$= \left(\frac{z(t)-z(t+h)}{-h}, \frac{y(t)-y(t+h)}{-h}, \frac{x(t)-x(t+h)}{-h}\right) = (z', y', x')$$

Similarly, $\lim_{h\to 0}\frac{f(t-h)\ominus f(t)}{-h} = (z', y', x')$

2.7 Existence of several solutions under Hukuhara differentiability:

In the following equivalent crisp differential equations are considered

$$u' = -u + \sigma(t), u' - \sigma(t) = -u \; and \; u' + u = \sigma(t), \qquad u(0) = u_0$$

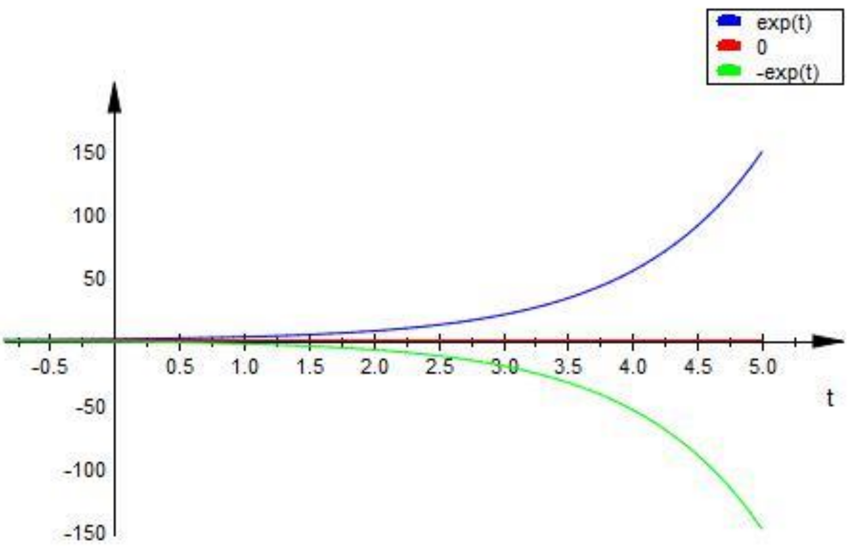
When these equations are fuzzified we get three different fuzzy differential equations and exhibit very different behaviors. In this section, we begin with the inequivalent homogeneous FIVPs, and then contrast their behavior with the behavior of the solutions of the corresponding nonhomogeneous FDEs. In this section we use exclusively the Hukuhara type differentiability.

Example:

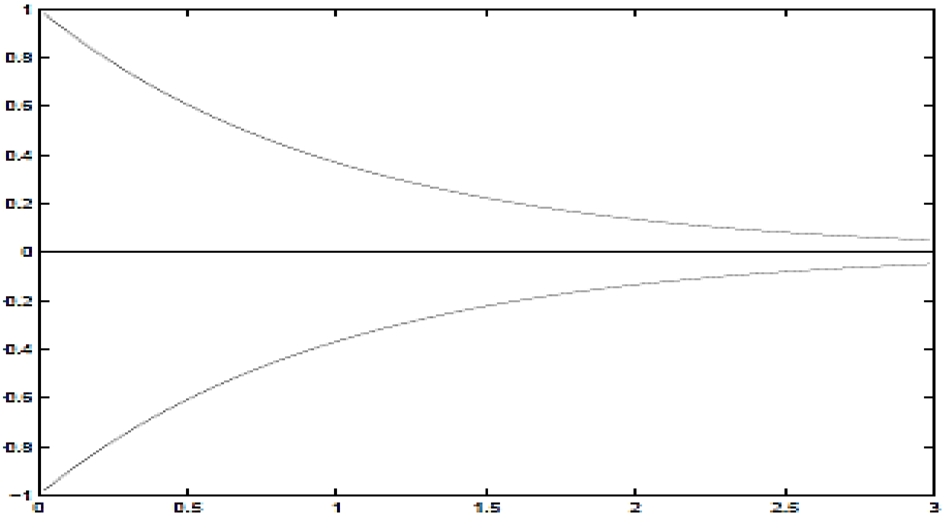
Let us consider FIVP

$$u' = -u, u(0) = (-1,0,1)$$

The solution of this problem is $u(t) = (-e^t, 0, e^t)$. Its graphical representation is



Surely since (i)-differentiability is in fact Hukuhara differentiability we obtain the unstable solution of the above figure. Under (ii)-differentiability condition we get the solution $u(t) = e^{-t}(-1,0,1)$. Solution of above equation under (ii)-differentiability is represented in the below figure.



Now, if we consider the corresponding equivalent nonhomogeneous FIVP,

$$u' + u = 2e^{-t}(-1,0,1), u(0) = (-1,0,1) \qquad (1)$$

$$u' = -u + 2e^{-t}(-1,0,1), u(0) = (-1,0,1) \qquad (2)$$

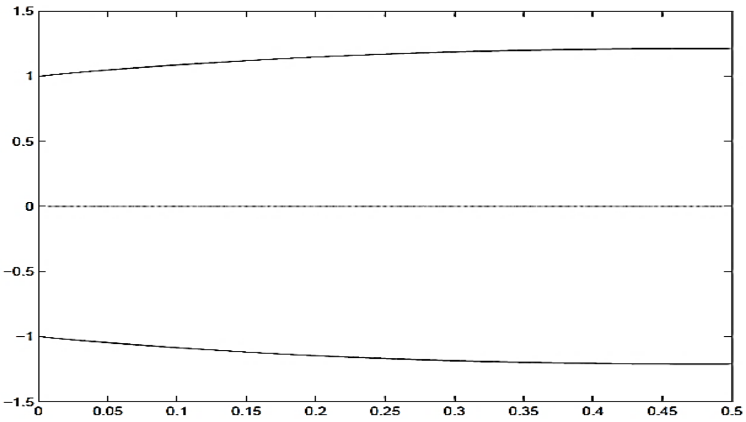
$$u' - 2e^{-t}(-1,0,1) = -u, u(0) = (-1,0,1) \qquad (3)$$

Now from (1),we get

$$u' + u = 2e^{t}\left(-1,0,1\right), u\left(0\right) = \left(-1,0,1\right)$$

for x	for y	for z
$x' + x = -2e^{-t}; x(0) = -1$	$y' + y = 0; y(0) = 0$	$z' + z = 2e^{-t}; z(0) = 1$
$x(t) = -(2t+1)e^{-t}$	$y(t) = 0$	$z(t) = (2t+1)e^{-t}$

$$\therefore u(t) = [x(t), y(t), z(t)] = [-(2t+1)e^{-t}, 0, (2t+1)e^{-t}]$$



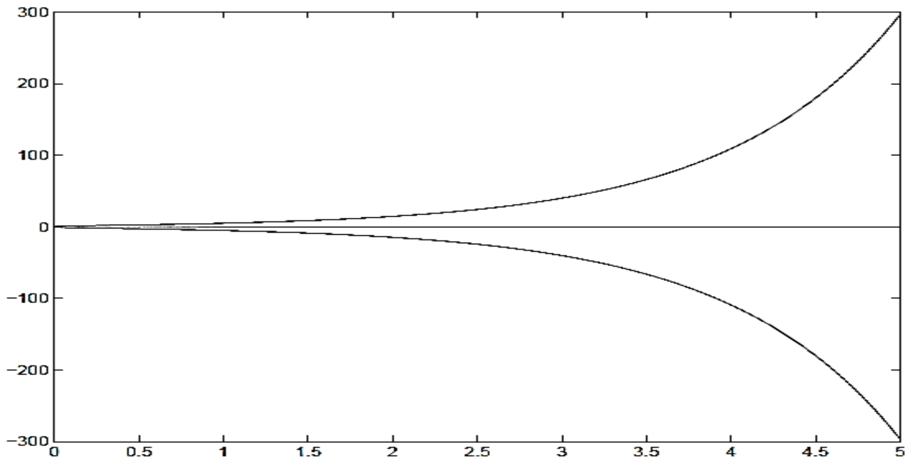
So, we get that u is a solution of (1) on [0,0.5],we consider it on this interval only, however it exists on [-0.5,0.5].

We now demonstrate the behavior of the solution when we consider the FIVP (2) in a different formulation.

$$u' = -u + 2e^{-t}\left(-1,0,1\right), u\left(0\right) = \left(-1,0,1\right)$$

for x	*for y*	*for* z
$x' = -z - 2e^{-t}; x(0) = -1$	$y' = -y; y(0) = 0$	$z' = -x + 2e^{-t}; z(0) = 1$
$x(t) = -(2t+1)e^{-t}$	$y(t) = 0$	$z(t) = (2t+1)e^{-t}$

$$\therefore u(t) = [x(t), y(t), z(t)] = [e^{-t} - 2e^{-t}, 0, 2e^{-t} - e^{-t}], t \in (0, \infty)$$



Let us consider the last equation on the list of inequivalent FDEs (3) obtained from equivalent crisp ODE

$u' - 2e^{-t}(-1,0,1) = -u, u(0) = (-1,0,1)$

for x	*for y*	*for z*
$x' - 2e^{-t} = -z; x(0) = -1$	$y' = -y; y(0) = 0$	$z' + 2e^{-t} = -x; z(0) = 1$
$x(t) = -e^{-t}$	$y(t) = 0$	$z(t) = e^{-t}$

$\therefore u(t) = [x(t), y(t), z(t)] = [-e^{-t}, 0, e^{-t}]$

but in this case u is not H-differentiable since the H-differences $u(t+h) \ominus u(t)$ and $u(t) \ominus u(t-h)$ do not exist.

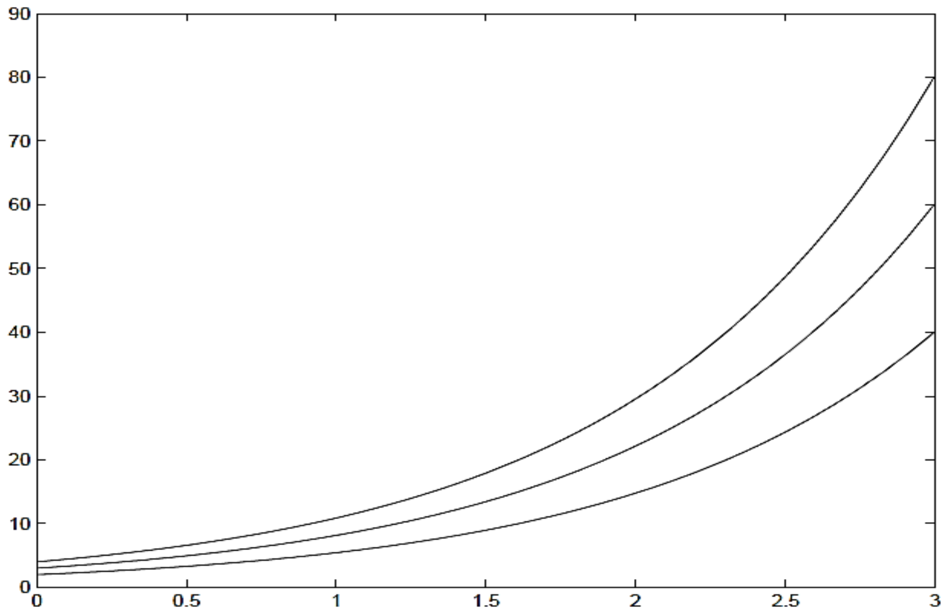
We observe that the solutions of the equations (1) and (2) behave in quite different ways, as shown in Figures ,however these equations are different fuzzyfications of equivalent crisp ODEs.

Now we consider another example in two different formulations to see illustrate the situation.

Firstly let us consider the homogeneous FDE,

$$u' = u, u(0) = (2,3,4)$$

It is easy to check that $u(t) = e^{t}(2,3,4)$ is Hukuhara differentiable solution of the above equation over [0;1). This solution is illustrated in the following Figure



Consider the initial value problems

$$u' = u + (1,2,3)t, u(0) = (2,3,4) \quad (4)$$

$$u' + (-1)(1,2,3)t = u, u(0) = (2,3,4) \quad (5)$$

$$u' - u = (1,2,3)t, u(0) = (2,3,4) \quad (6)$$

Now from (4),

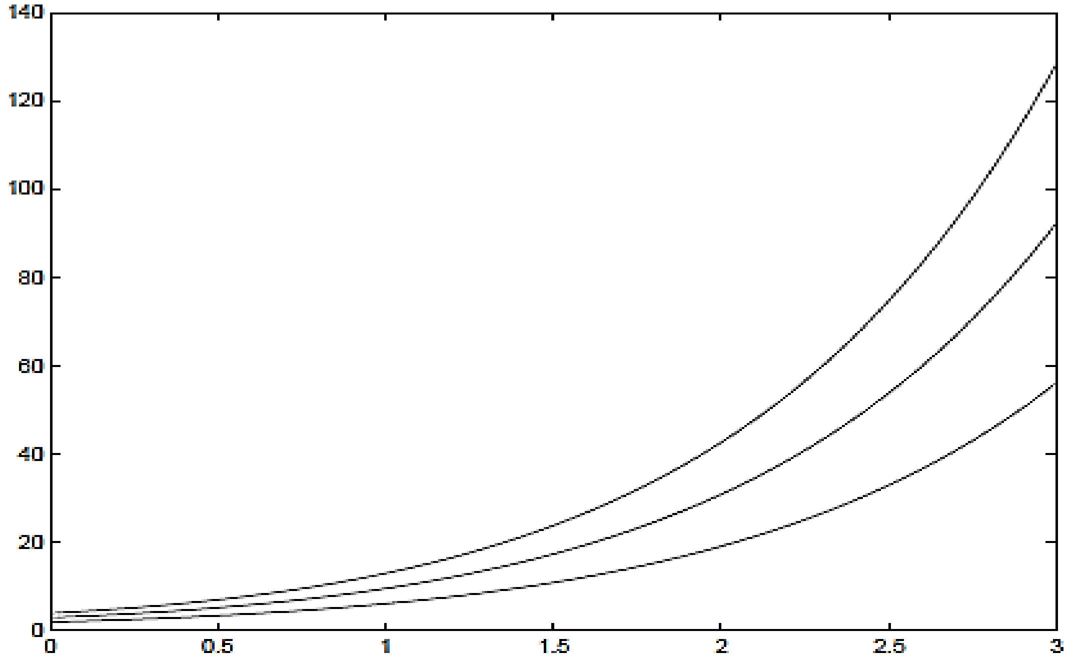
$$u' = u + (1,2,3)t, u(0) = (2,3,4)$$

$for\ x$	$for\ y$	$for\ z$
$x' = x + t; x(0) = 2$	$y' = y + 2t; y(0) = 3$	$z' = z + 3t; z(0) = 4$
$x(t) = 3e^t - t - 1$	$y(t) = 5e^t - 2t - 2$	$z(t) = 7e^t - 3t - 3$

$$\therefore u(t) = [x(t), y(t), z(t)] = [3e^t - t - 1, 5e^t - 2t - 2, 7e^t - 3t - 3], t \in [0, \infty)$$

is Hukuhara differentiable and it is a solution of (4)

It's graphical representation is



Now from (5),

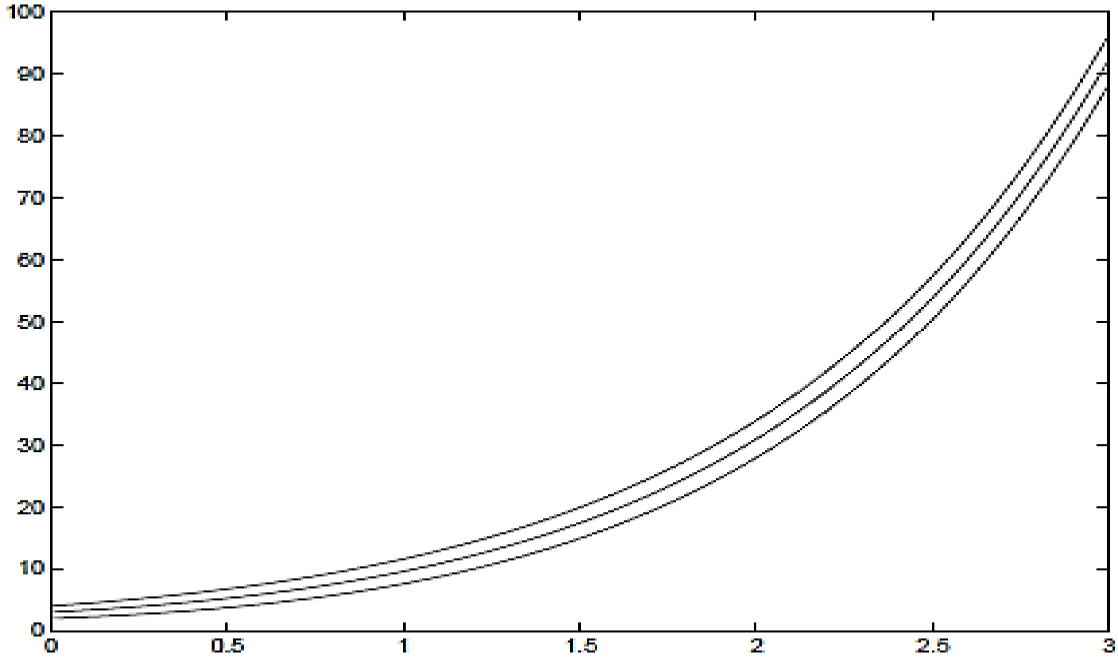
$$u' + (-1)(1,2,3)t = u, u(0) = (2,3,4)$$

for x	*for y*	*for z*
$x' - 3t = x; x(0) = 2$	$y' - 2t = y; y(0) = 3$	$z' - t = z; z(0) = 4$
$x(t) = 5e^t - 3t - 3$	$y(t) = 5e^t - 2t - 2$	$z(t) = 5e^t - t - 1$

$$\therefore u(t) = [x(t), y(t), z(t)] = [3e^t - t - 1, 5e^t - 2t - 2, 7e^t - 3t - 3], t \in [0, \infty)$$

is Hukuhara differentiable and it is a solution of (5)

It's graphical representation is shone in the next picture



Now from (6),

$$u' - u = (1,2,3)t, u(0) = (2,3,4)$$

for x	for y	for z
$x' - z = t; x(0) = 2$	$y' - y = 2t; y(0) = 3$	$z' - x = 3t; z(0) = 4$
$x(t) = 5e^{t} - 3t - 2e^{-t} - 1$	$y(t) = 5e^{t} - 2t - 2$	$z(t) = 5e^{t} - t + 2e^{-t} - 3$

$$\therefore u(t) = [x(t), y(t), z(t)] = [5e^{t} - 3t - 2e^{-t} - 1, 5e^{t} - 2t - 2, 5e^{t} - t + 2e^{-t} - 3], t \in (\ln 2, \infty)$$

Since this is not a solution near the origin we do not consider it a proper solution of the problem (6).

$\therefore$ *The graphical representation of the solutions of* (4) *and* (5) *can be seen in the above figer respectively. Again we have quite different behavior. Indeed, for the second solution we could say that it is "relatively stable", i.e. the uncertainty is quite small w.r.t. the core for large values of t.*

A PREY PREDATOR MODEL WITH FUZZY INITIAL VALUES

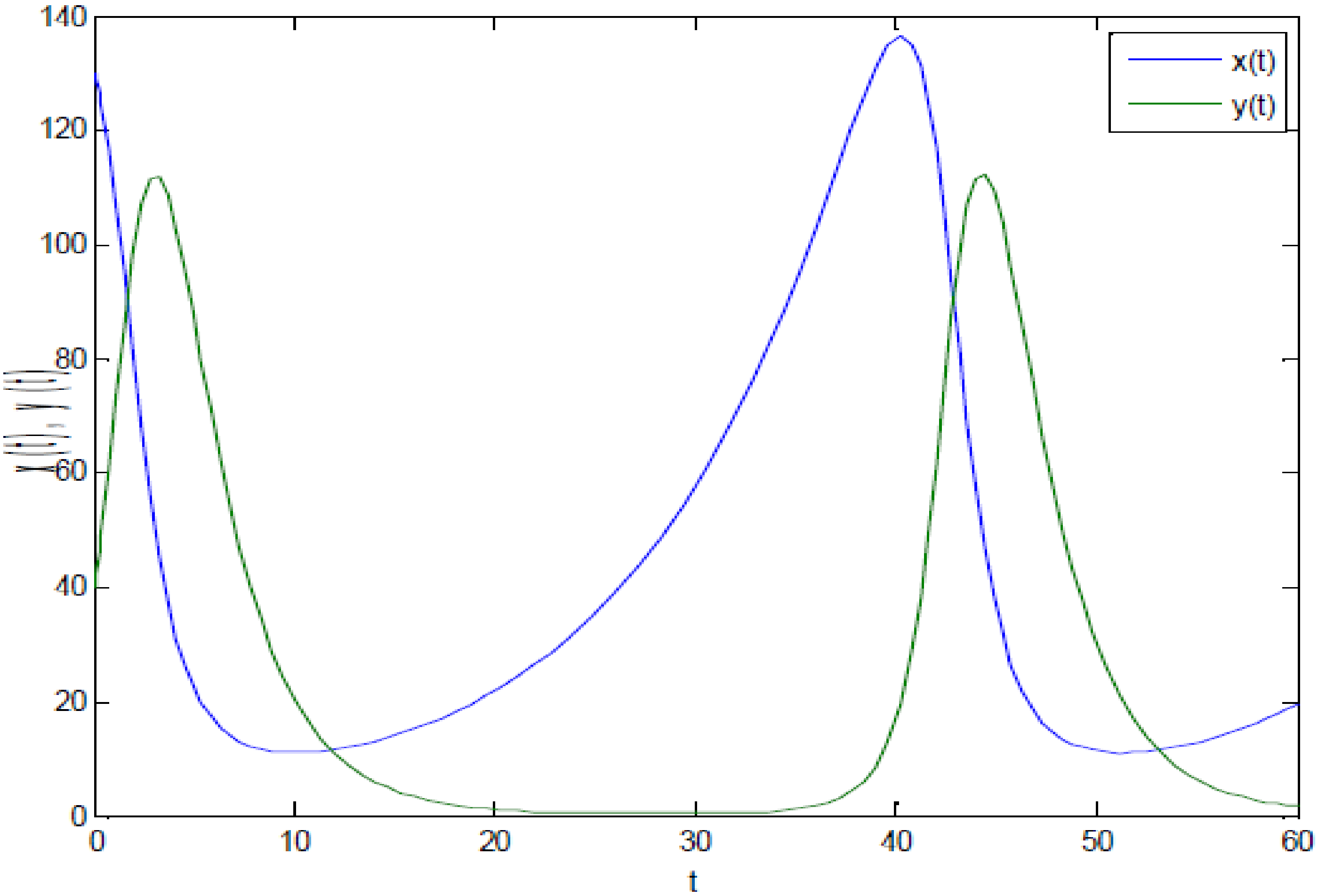
Here we discuss the prey predator model with the help of an example.

Consider the following prey-predator model with fuzzy initial values. Before giving a solution of the fuzzy problem we want to find its crisp solution.

$$\left.\begin{aligned}\frac{dx}{dt} &= 0.1x - 0.005xy \\ \frac{dy}{dt} &= -0.4y + 0.008xy\end{aligned}\right\} \text{with initial condition } x(0) = 130,\ y(0) = 40$$

where x(t) and y(t) are the number of preys and predators at time t, respectively.

Crisp solutions for the above problem are given in the below Figure



Let the initial values be fuzzy i.e $x(0) = \widetilde{130}$ *and* $y(0) = \widetilde{40}$ and let their α-level sets be as follows

$$x(\alpha) = \left[130\right]^{\alpha} = [100 + 30\alpha, 160 - 30\alpha]$$

$$y(\alpha) = \left[40\right]^{\alpha} = [20 + 20\alpha, 60 - 20\alpha]$$

Let the α-level sets of $x(t,\alpha)$ be $[x(t,\alpha)]^{\alpha} = [u(t,\alpha), v(t,\alpha)]$, and for simplicity denote them as $[u, v]$, similarly $[y(t,\alpha)]^{\alpha} = [r(t,\alpha), s(t,\alpha)] = [r, s]$.
Then

$$[u', v'] = 0.1[u, v] - 0.005[u, v].[r, s],$$

$$[r', s'] = -0.4[r, s] + 0.008[u, v].[r, s]$$

Hence for α= 0 the above initial value problem derives

1) If $x(t,\alpha)$ and $y(t,\alpha)$ are (i)-differentiable then the above problem becomes

$$u' = 0.1u - 0.005vs$$

$$v' = 0.1v - 0.005ur$$

$$r' = -0.4s + 0.008ur$$

$$s' = -0.4r + 0.008vs$$

with $u(0) = 100, v(0) = 160, r(0) = 20, s(0) = 60$

2) If $x(t,\alpha)$ and $y(t,\alpha)$ are (ii)-differentiable then the above problem becomes

$$v' = 0.1u - 0.005vs$$

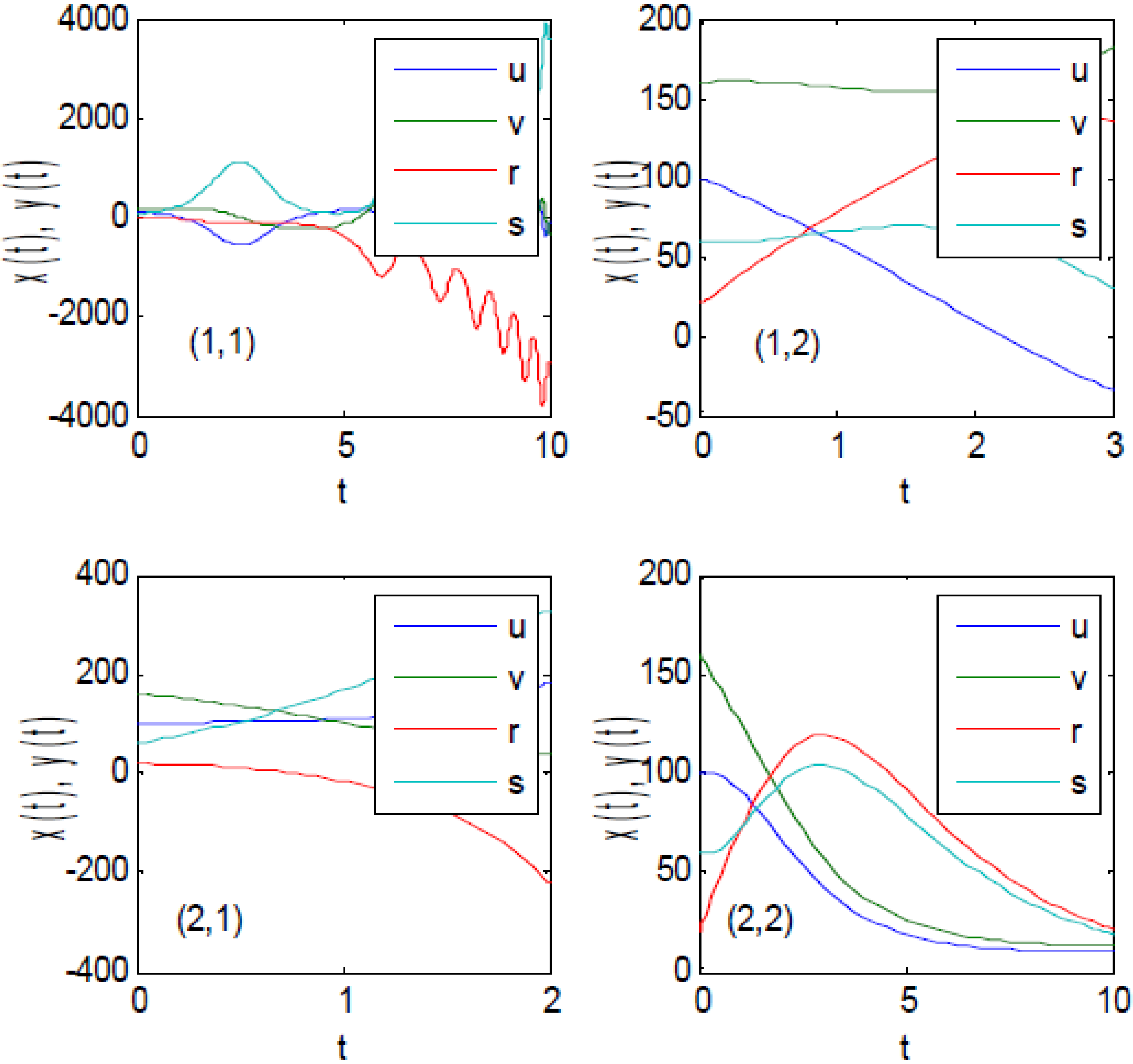
$$u' = 0.1v - 0.005ur$$

$$s' = -0.4s + 0.008ur$$

$$r' = -0.4r + 0.008vs$$

with $u(0) = 100, v(0) = 160, r(0) = 20, s(0) = 60$

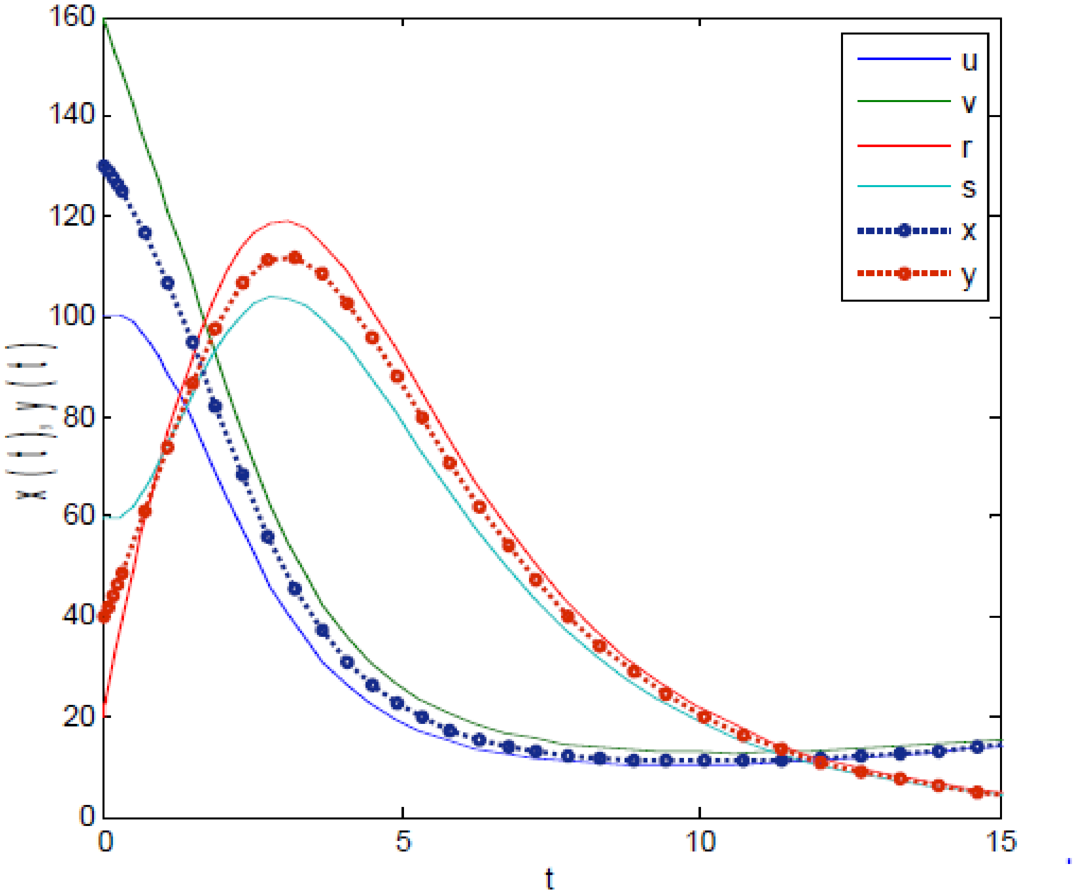
Now for $\alpha = 0$ the graphical solution of all possible cases are given below
In the below picture ,
(1,1) means that $x(t,\alpha)$ and $y(t,\alpha)$ are (i)-differentiable,
(1,2) means that x(t,α) is (i)-differentiable and y(t,α) is (ii)-differentiable,
(2,1) means that x(t,α) is (ii)-differentiable and y(t,α) is (i)-differentiable,
(2,2) means that x(t,α) and y(t,α) are (ii)-differentiable,



Fuzzy solutions of the Problem for $\alpha = 0$

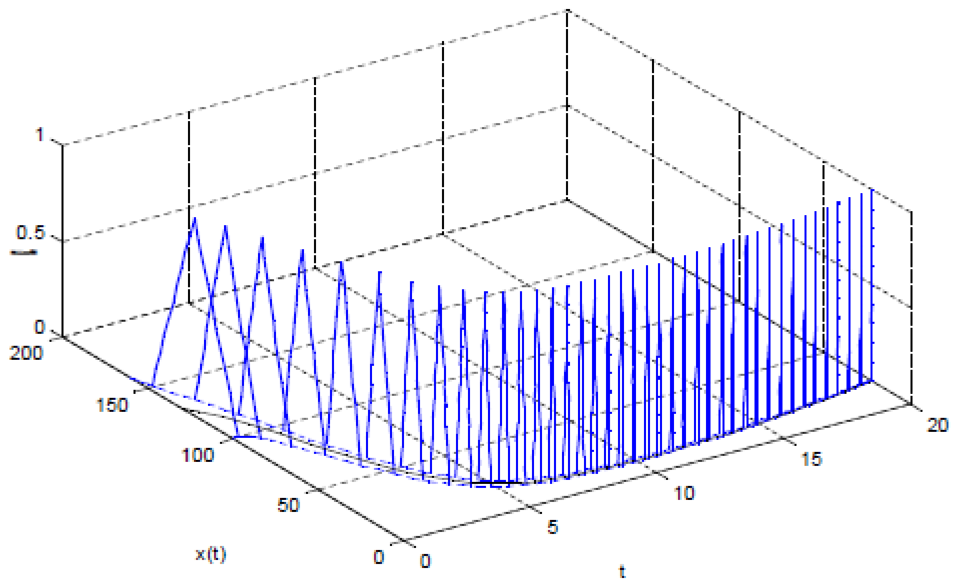
Now if we analyze the above Figure, we observe that when $x(t,\alpha)$ and $y(t,\alpha)$ are (2)-differentiable the graphical solution is biologically meaningful, furthermore the graphical solution is coherent with the crisp solution. On the contrary, when $x(t,\alpha)$ and $y(t,\alpha)$ are differentiable as (1,1), (1,2), (2,1) the graphical solutions are incompatible with biological facts.

So we focus on the situation when $x(t,\alpha)$ and $y(t,\alpha)$ are (ii)-differentiable. We give the crisp graphical solution and fuzzy graphical solution when $x(t,\alpha)$ and $y(t,\alpha)$ are (ii)-differentiable on the same graph for $\alpha = 0$ and $\alpha \in [0,1]$. The crisp solution and fuzzy solution for $\alpha = 0$ are given in the below Figure.

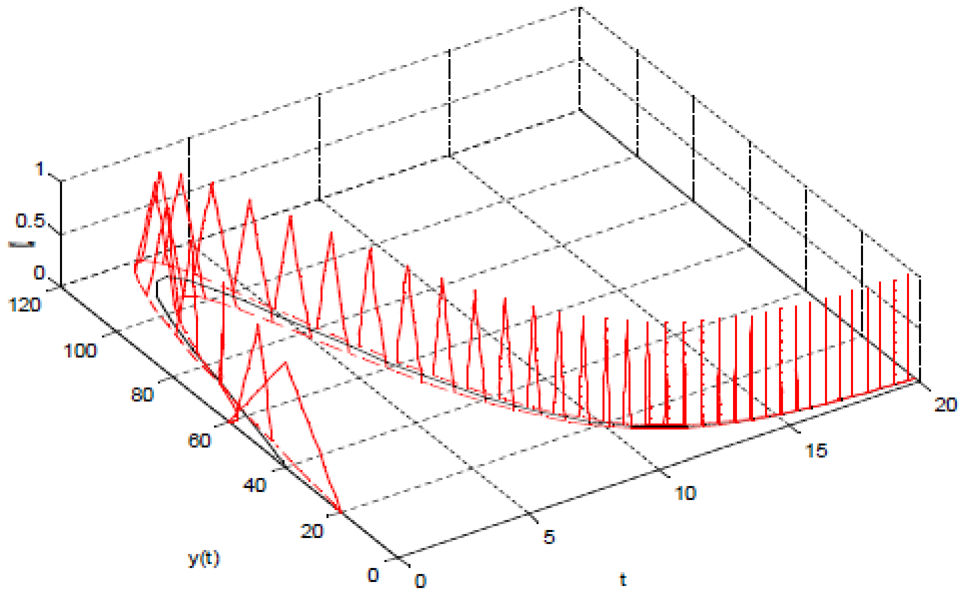


Crisp solution and fuzzy solution for $\alpha = 0$

In below two Figures, if we set $\alpha = 0$, we see the crisp solution confined by the left and right branches of the dependent variables x(t), y(t). For example in Figure, x(t) is confined by u and v for α = 0. Additionally, if we set α =1 the projection of the peaks of the triangles coincides with the crisp solution.



$x(t)\ for\ \alpha \in [0, 1]$



$y(t) for\ \alpha \in [0, 1]$

As we see in this example, the uniqueness of the solution of a fuzzy initial value problem is lost when we use the strongly generalized derivative concept. This situation is looked on as a disadvantage. Researchers can choose the best solution which better reflects the behavior of the system under consideration, from multiple solutions.

Conclusions And Future Work

Surely, the multitude of solutions that are obtained is not really a disadvantage, since from all the solutions we can find those which better reflect the behavior of the system under study. This selection of the best solution in our opinion can be made only from an accurate study of the physical properties of the system which is studied. This makes it necessary to study fuzzy differential equations as an independent discipline, and exploring it further in different directions to facilitate its use in modeling entirely different physical and engineering problems satisfactorily. In this sense, the different approaches are complimentary to each other.

As we see in this work, the uniqueness of the solution of a fuzzy initial value problem is lost when we use the strongly generalized derivative concept. This situation is looked on as a disadvantage, but actually it is not because researchers can choose the best solution which better reflects the behavior of the system under consideration, from multiple solutions. Here a question may arise. Which solution is the best? The answer for this may come after a precise analysis of the physical properties of the system which is under study.

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